Ch4: Equilibrium of Rigid Bodies

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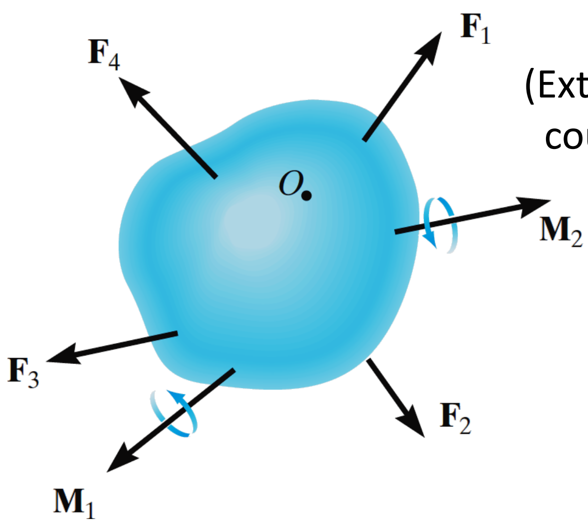
Equilibrium and Free-Body Diagrams

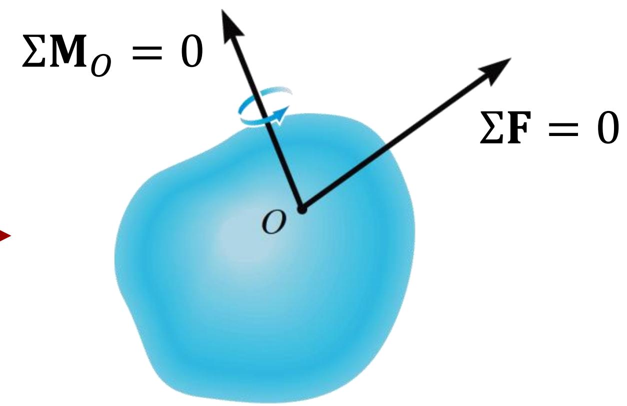
Equilibrium Conditions

If the **resultant force and couple moment** (at any arbitrary point O on or off the body) are both equal to **zero**, then the body is said to be in **Equilibrium**.

$$\begin{cases} \Sigma \mathbf{F} = \mathbf{0} \\ \Sigma \mathbf{M}_O = \mathbf{0} \end{cases}$$



(External forces and couple moments)

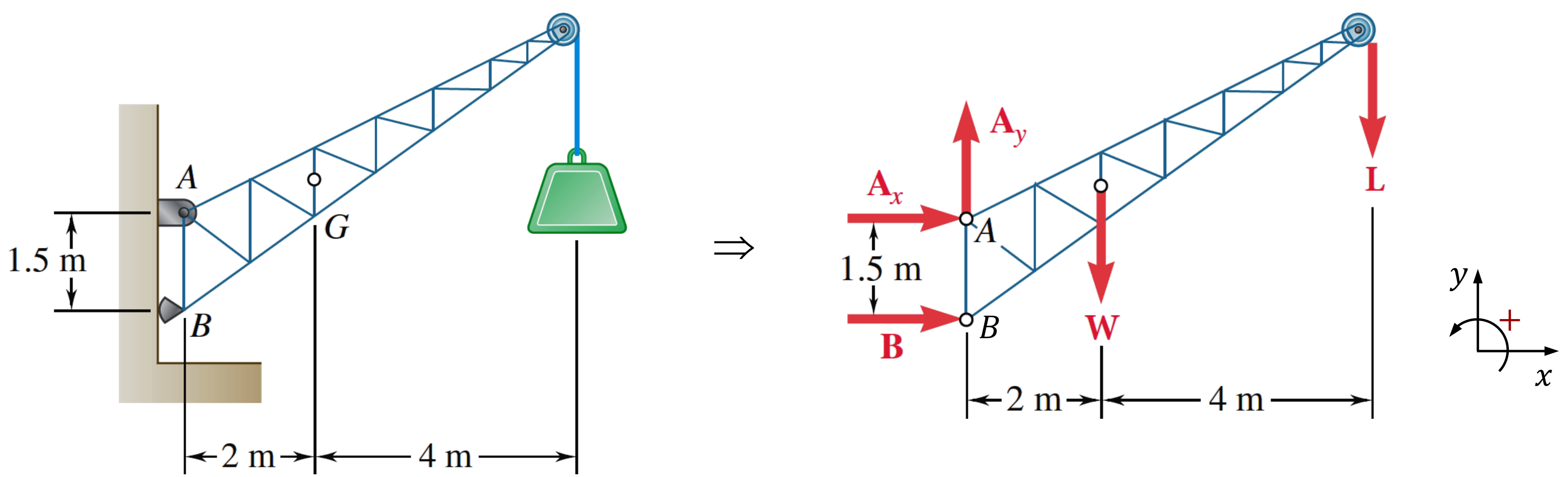


We can use these equations to determine unknown forces **applied** to the rigid body or unknown **reactions** exerted on it by its supports.

Free-Body Diagrams

In order to write the equations of equilibrium for a rigid body, we must first draw a **free-body diagram** (**FBD**) of the rigid body under consideration.

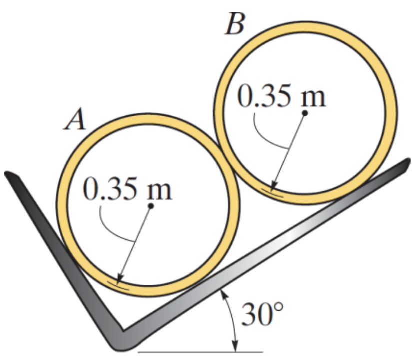
This diagram is an **isolated** sketch of the body from its surroundings. The free-body diagram should include **all the external forces and couple moments** that the surroundings exert on the body (including the reactions exerted on it by its supports, and the weight of the body) and **dimensions** (for computing moments of forces).



Free-Body Diagrams

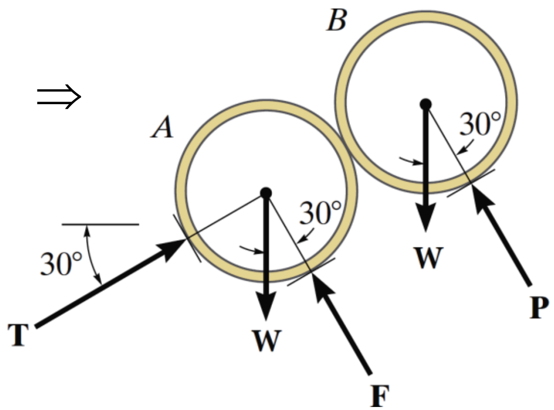
- If the body under consideration is made of several parts, do not include the **internal forces** the various parts exert on each other among the external forces, since they occur in equal but opposite collinear pairs (Newton's third law) and therefore cancel out.

- Include the weight **W** of the body, acting at the **center of gravity** G of the body, among the external forces. When the body is **uniform**, the center of gravity will be located at the body's **geometric center** or **centroid**.

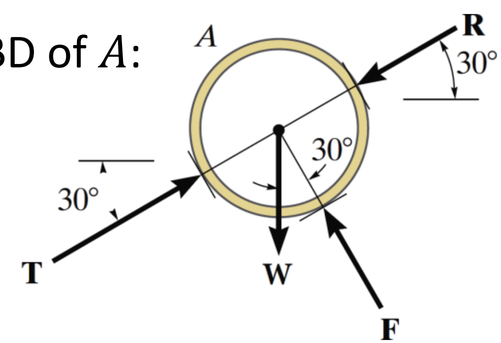


⇒

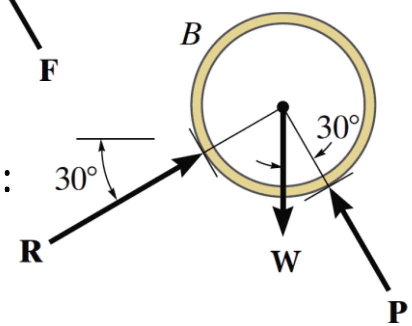
FBD of A, B:



FBD of A:



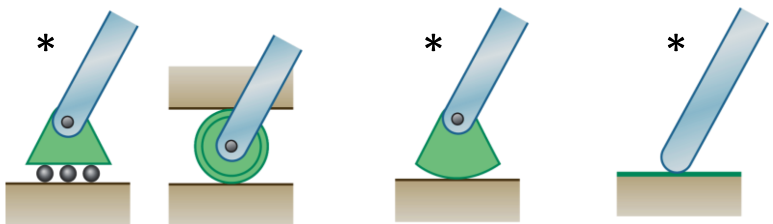
FBD of B:



Equilibrium in Two Dimensions

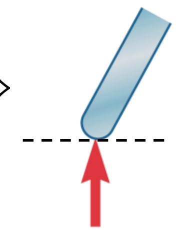
Support Reactions

- Reactions with **one unknown** (the magnitude is known, and line of action is unknown).

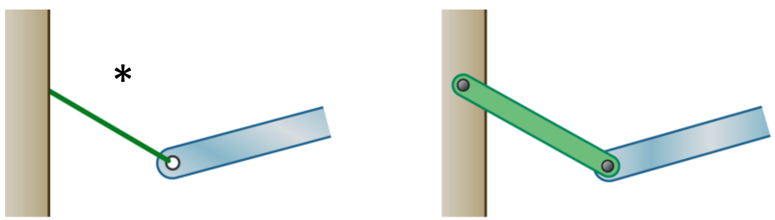


Rollers Rocker Frictionless Surface

$\Rightarrow$

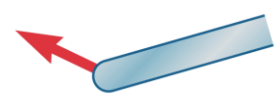


Force is perpendicular to surface.

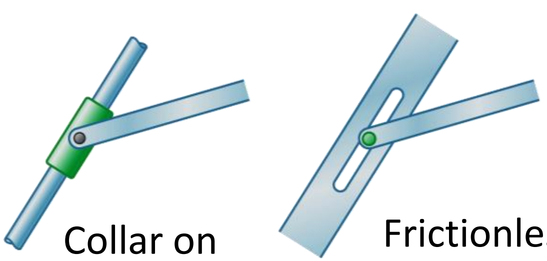


Weightless Cable Weightless Link

$\Rightarrow$



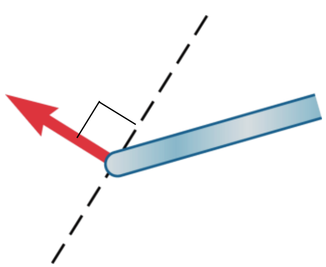
Force is along cable or link.



Collar on frictionless rod

Frictionless pin in slot

$\Rightarrow$

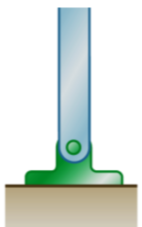


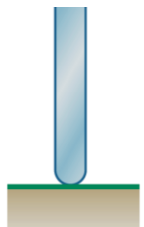
Force is perpendicular to rod or slot.

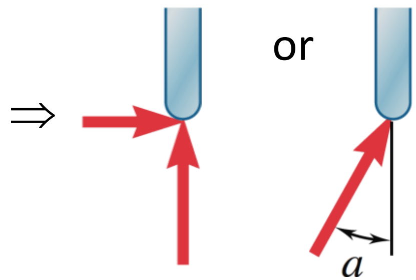
(* the sense is also known)

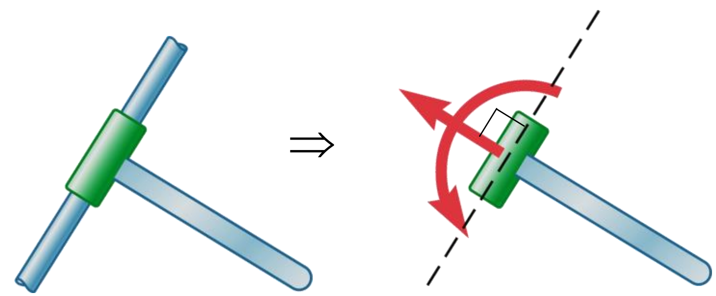
Support Reactions

- Reactions with **two unknowns**.


Frictionless pin or hinge

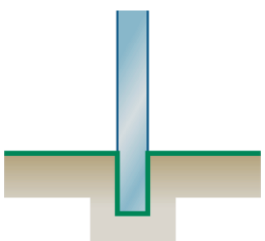

Rough surface

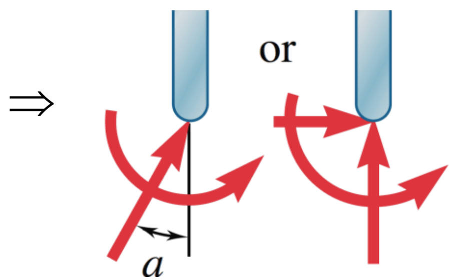




Member fixed connected to collar on smooth rod

- Reactions with **three unknowns**.


Fixed support



A support prevents the translation/rotation of a body in a given direction by exerting a force/couple moment on the body in the opposite direction.

Note: When the sense of an unknown force or couple is unknown, arbitrarily assume the sense; the <u>sign of the answer</u> will indicate whether the assumption is correct or not.

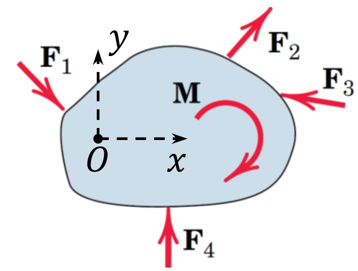
A positive [negative] answer means the assumption is correct [incorrect].

Equations of Equilibrium

If the force system acting on a rigid body lies in a single plane (say $x - y$) and any couple moments acting on the body are directed perpendicular to this plane, the force and couple system is **two-dimensional**. Equations of equilibrium:

$$\Sigma F_x = 0, \quad \Sigma F_y = 0, \quad \Sigma M_O = 0$$

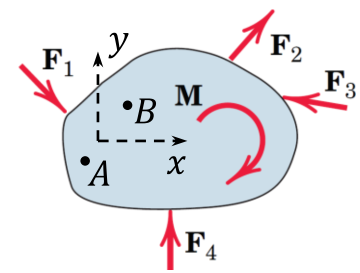
(O is any point in the plane of the body)



- Two alternative sets of three independent equilibrium equations:

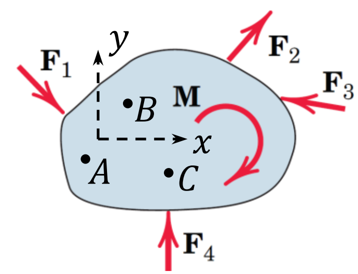
$$\Sigma F_{\underbrace{x}_{\text{or } y}} = 0, \quad \Sigma M_A = 0, \quad \Sigma M_B = 0$$

Only when the line AB is not perpendicular to the $\underbrace{x}_{\text{or } y}$ axis.



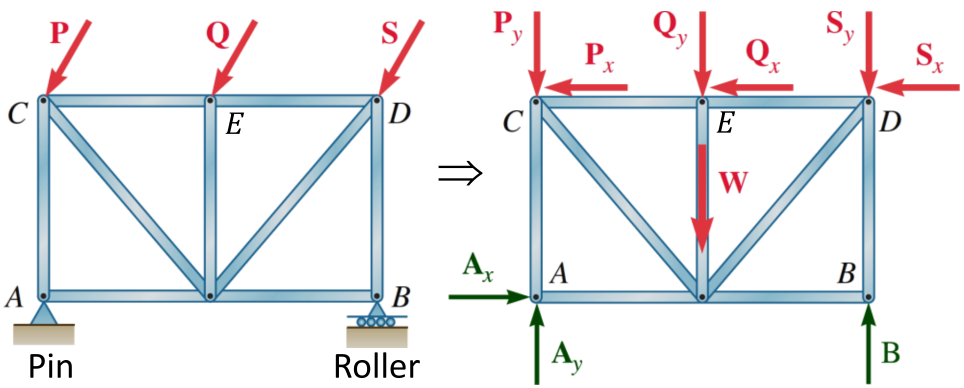
$$\Sigma M_A = 0, \quad \Sigma M_B = 0, \quad \Sigma M_C = 0$$

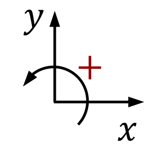
Only when the points A, B, and C not lie along a straight line.



These three equations can be solved for **up to three unknowns**.

Equations of Equilibrium





For given **P**, **Q**, **S**, and **W**:

$\Sigma M_A = 0 \rightarrow B$ ✓ (It doesn't contain A_x , A_y)

$\Sigma F_x = 0 \rightarrow A_x$ ✓ (It doesn't contain A_y , B)

$\Sigma F_y = 0 \rightarrow A_y$ ✓

✦ It is desirable to choose equations of equilibrium containing **only one unknown**, because this eliminates the necessity of solving simultaneous equations. To do so,

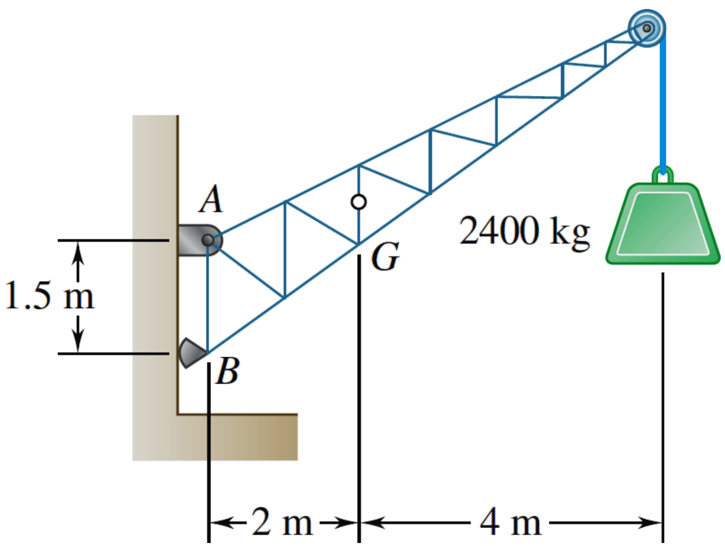
i. Sum moments about the <u>point of intersection</u> of the lines of action of two unknown forces (for example, point E is not a proper choice).

ii. Sum force components in <u>a direction perpendicular</u> to two unknown parallel forces.

✦ Any additional equation (e.g., $\Sigma M_B = 0$) <u>is not independent</u> (does not contain any new information) and cannot be used to determine a fourth unknown (however, they can be useful for <u>checking the solution</u> obtained from the original three equations of equilibrium!).

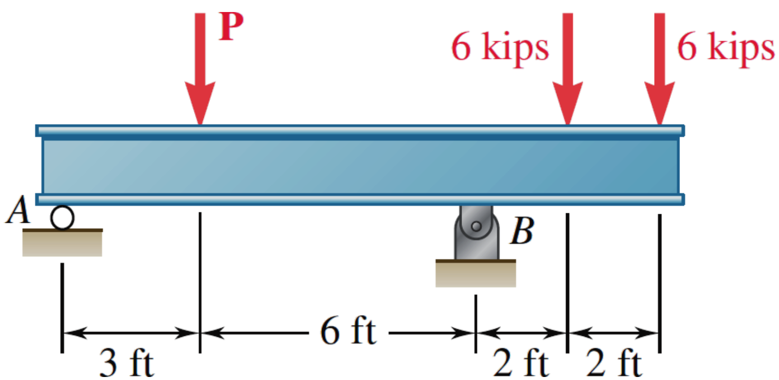
Sample Problem 4.1

A fixed crane has a mass of 1000 kg and is used to lift a 2400-kg crate. It is held in place by a pin at A and a rocker at B. The center of gravity of the crane is located at G. Determine the components of the reactions at A and B.



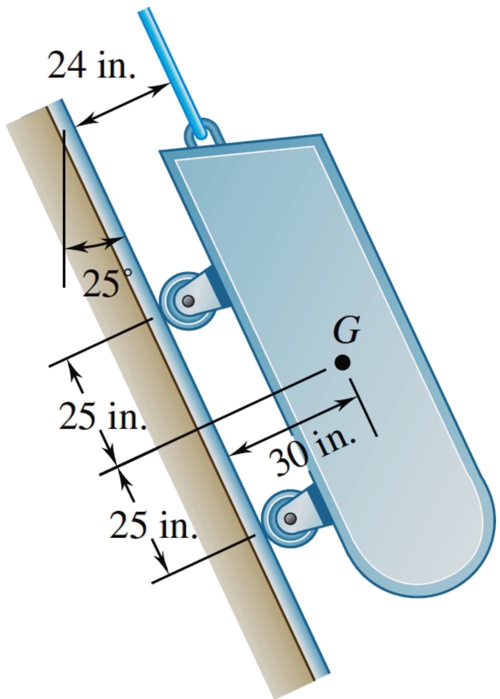
Sample Problem 4.2

Three loads are applied to a beam as shown. The beam is supported by a roller at A and by a pin at B. Neglecting the weight of the beam, determine the reactions at A and B when $P = 15$ kips.



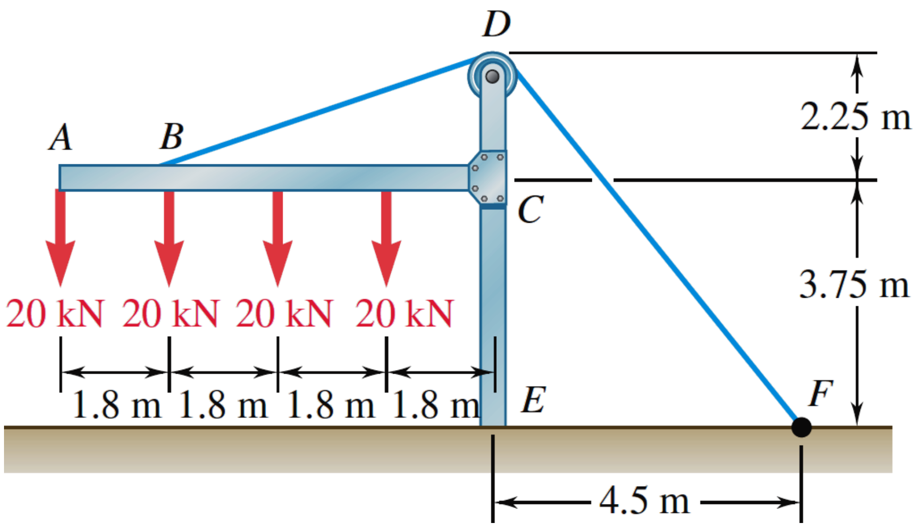
Sample Problem 4.3

A loading car is at rest on a track forming an angle of 25° with the vertical. The gross weight of the car and its load is 5500 lb, and it acts at a point 30 in. from the track, halfway between the two axles. The car is held by a cable attached 24 in. from the track. Determine the tension in the cable and the reaction at each pair of wheels.



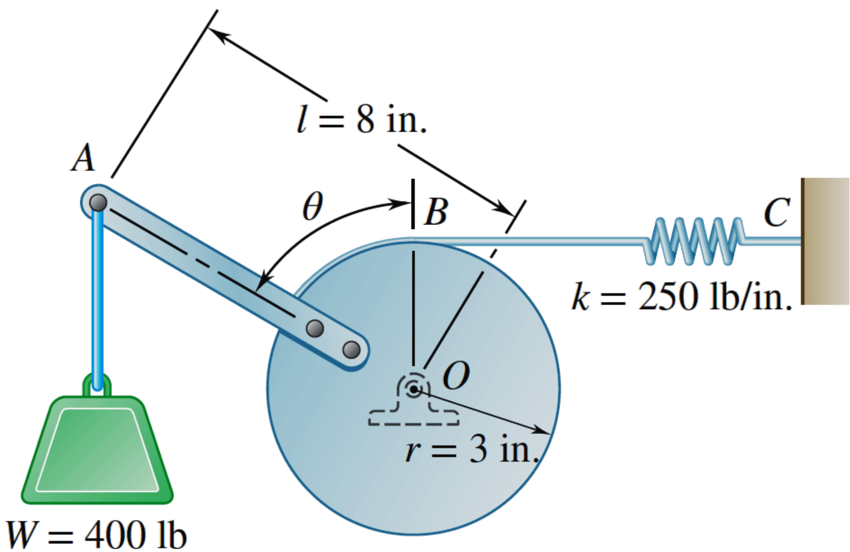
Sample Problem 4.4

The frame shown supports part of the roof of a small building. Knowing that the tension in the cable is 150 kN, determine the reaction at the fixed end E.



Sample Problem 4.5

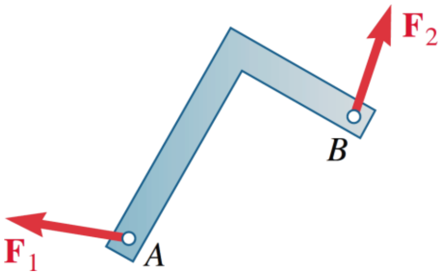
A 400-lb weight is attached at A to the lever shown. The constant of the spring BC is $k = 250$ lb/in., and the spring is unstretched when $\theta = 0$. Determine the position of equilibrium.



Two Special Cases

Equilibrium of a Two-Force Body

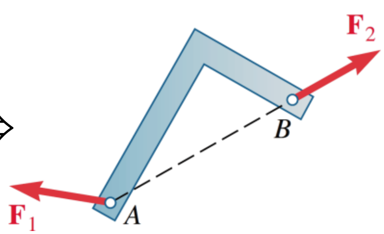
A **two-force body** is defined as a rigid body subjected to **forces acting at only two points**.



If a **two-force body is in equilibrium**, the two forces must have the **same magnitude**, **opposite sense**, and the **same line of action** (directed along the line joining the two points where these forces act).

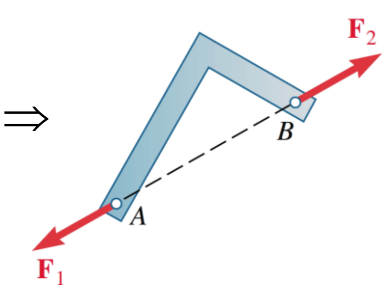
Proof: If a two-force body is in equilibrium:

$\Sigma M_A = 0 \quad \Rightarrow$



and

$\Sigma M_B = 0 \quad \Rightarrow$

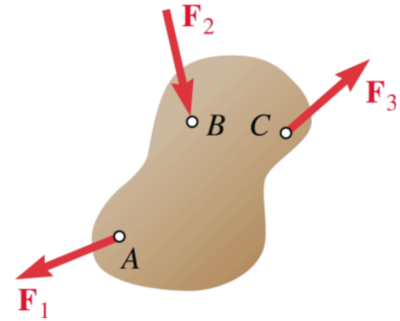


(the line of action of $\mathbf{F}_2$ must pass through A)

(the line of action of $\mathbf{F}_1$ must pass through B)

Equilibrium of a Three-Force Body

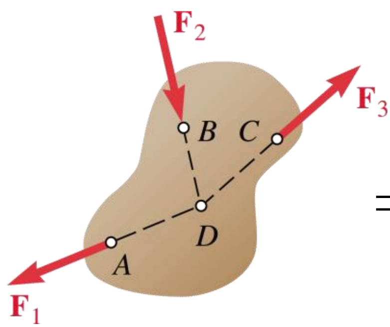
A **three-force body** is defined as a rigid body subjected to **forces acting at only three points**.



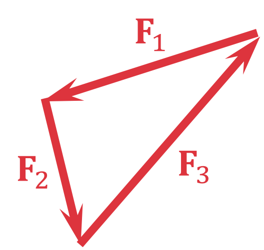
If a **three-force body is in equilibrium**, the lines of action of the three forces must be either **parallel** or **concurrent** (in this case, we can use the **triangle rule**).

Proof: If a three-force body is in equilibrium:

Case 1: If the lines of action of $\mathbf{F}_2$ and $\mathbf{F}_3$ intersect at point D, then the line of action of $\mathbf{F}_1$ must also pass through point D so that $\Sigma M_D = 0$.

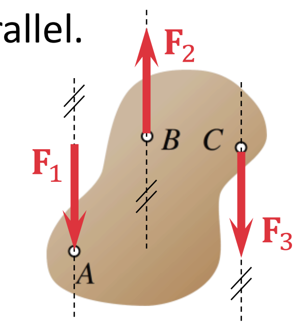


$\Rightarrow$



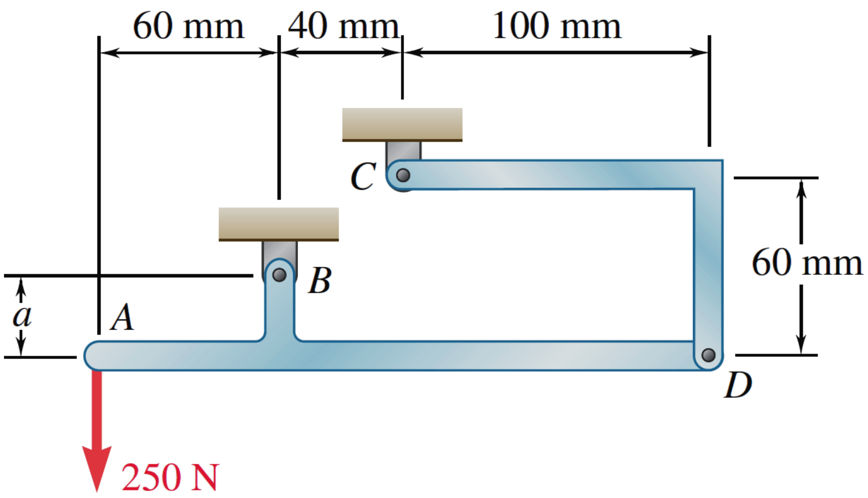
In this case, we can use the **triangle rule**.

Case 2: Lines of action are parallel.



Problem 4.65

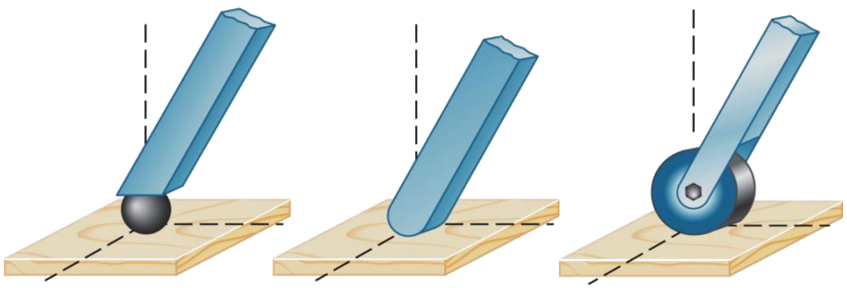
Determine the reactions at B and C when $a = 30$ mm.



Equilibrium in Three Dimensions

Support Reactions

Reactions with **one unknown**:



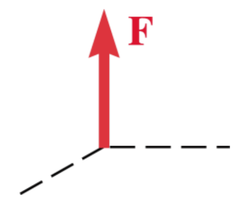
Ball

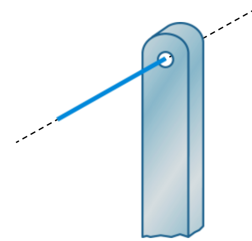
Frictionless surface

Flangeless wheel/roller

⇒

(perpendicular to surface)



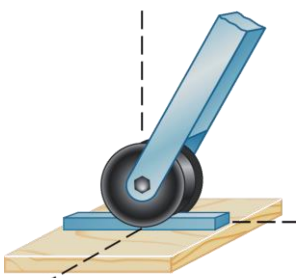


Cable

⇒

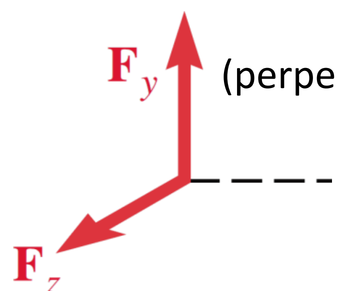
(along cable)

Reactions with **two unknowns**:



Wheel support with lateral constraint (flanged wheel)

⇒

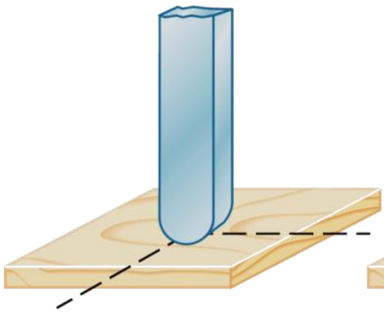


(perpendicular to surface)

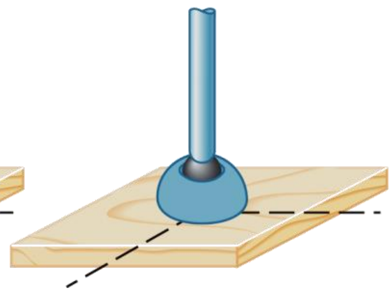
(parallel to axis of wheel)

Support Reactions

Reactions with **three unknowns**:

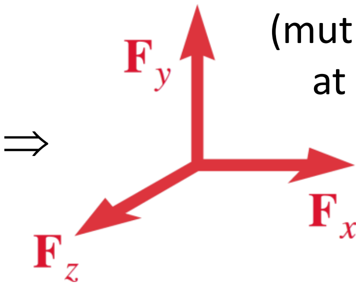


Rough surface



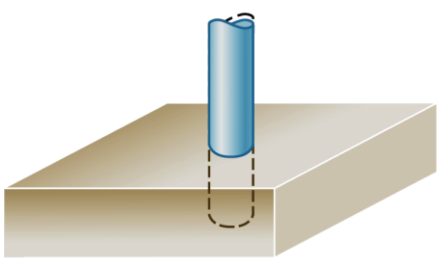
Ball and socket

⇒



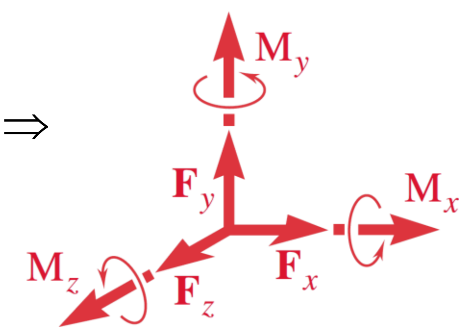
(mutually perpendicular at point of contact)

Reactions with **six unknowns**:



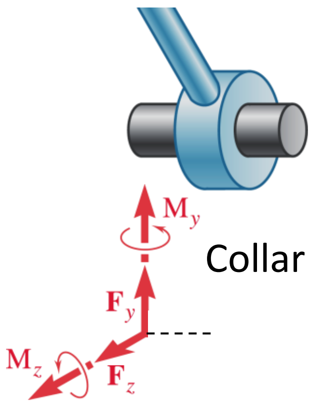
Fixed support

⇒

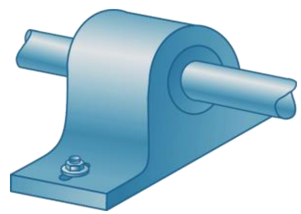


Support Reactions

Reactions with **four unknowns**:

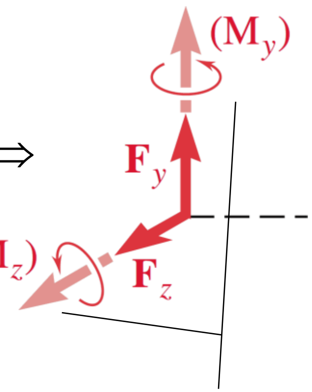


Collar



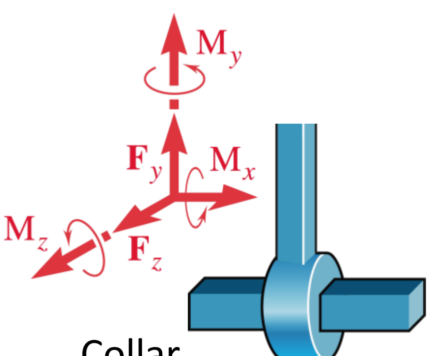
Bearing supporting radial load only

⇒

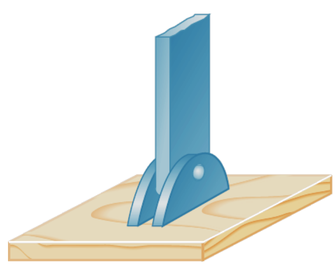


✦ They are primarily designed to prevent only translation. Thus, only force components should be included **unless** otherwise stated.

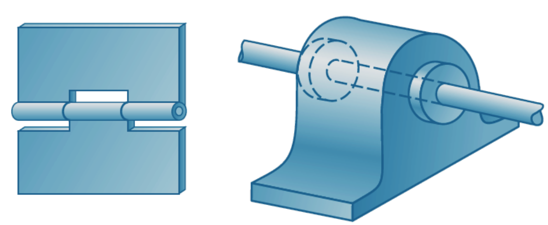
Reactions with **five unknowns**:



Collar

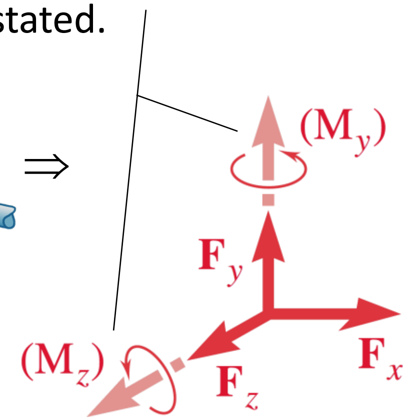


Pin and bracket



Hinge and bearing supporting axial thrust and radial load

⇒



Equations of Equilibrium

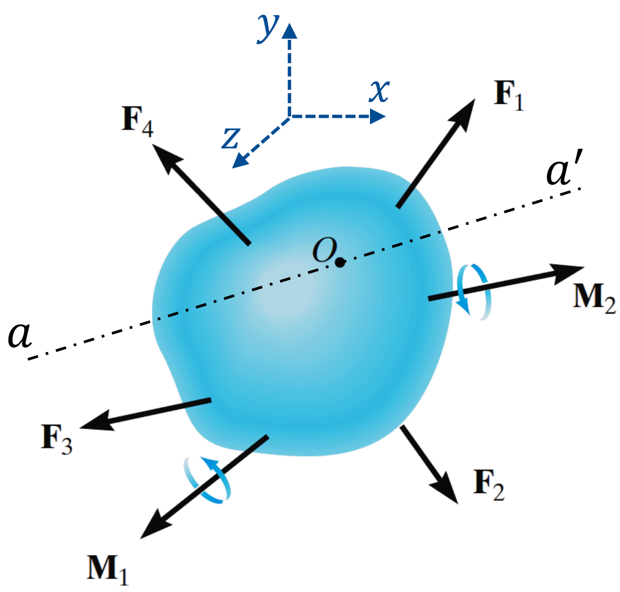
The most general situation of rigid-body equilibrium occurs in three dimensions.

$$\begin{cases} \Sigma \mathbf{F} = \mathbf{0} \\ \Sigma \mathbf{M}_O = \Sigma(\mathbf{r} \times \mathbf{F}) + \Sigma \mathbf{M}_i = \mathbf{0} \end{cases} \quad \text{(or)} \quad \begin{cases} \Sigma F_x = 0 & \Sigma F_y = 0 & \Sigma F_z = 0 \\ \Sigma M_x = 0 & \Sigma M_y = 0 & \Sigma M_z = 0 \end{cases}$$

(O is any point located either on or off the body)

$$\Sigma \mathbf{F} = \Sigma F_x \mathbf{i} + \Sigma F_y \mathbf{j} + \Sigma F_z \mathbf{k}$$

$$\Sigma \mathbf{M}_O = \Sigma M_x \mathbf{i} + \Sigma M_y \mathbf{j} + \Sigma M_z \mathbf{k}$$



✦ Any arbitrary **axis** aa' can be also chosen for summing moments:

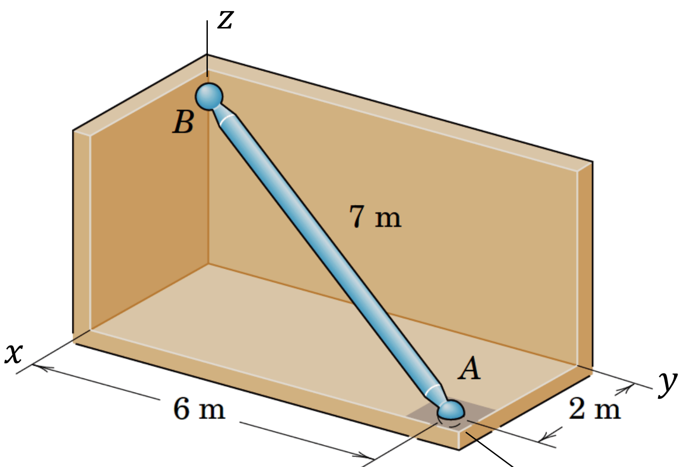
$$\Sigma \mathbf{M}_{aa'} = \mathbf{0}$$

These equations can be solved for **up to six unknowns**.

Equations of Equilibrium

✦ It is desirable to choose equations of equilibrium containing as few unknowns as possible. Two helpful strategies:

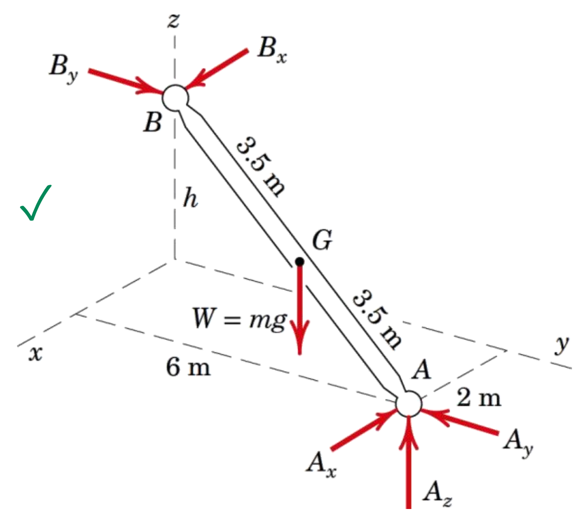
(1) Sum moments **about a support** to eliminate three unknown reaction components.



Ball and socket support

$$\Sigma \mathbf{M}_A = \mathbf{0} \quad \rightarrow \quad B_x, B_y \ \checkmark$$

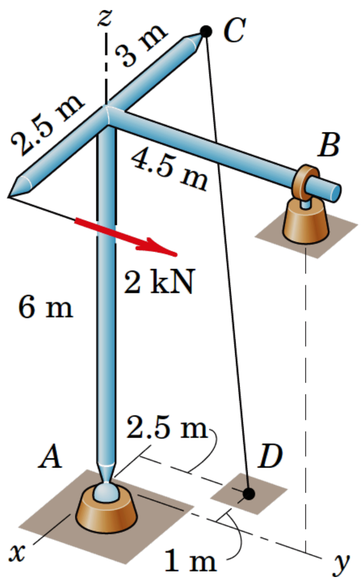
$$\Sigma \mathbf{F} = \mathbf{0} \quad \rightarrow \quad A_x, A_y, A_z \ \checkmark$$



Note: When the sense of an unknown force or couple is unknown, arbitrarily assume the sense. If the sign of the answer yields a **negative** scalar for a force or couple moment magnitude, it indicates that the correct sense is **opposite** to that assumed on the free-body diagram.

Equations of Equilibrium

(2) Sum moments **about an axis** that it intersects the lines of action of as many unknown forces as possible (say all but one of the unknown reactions).

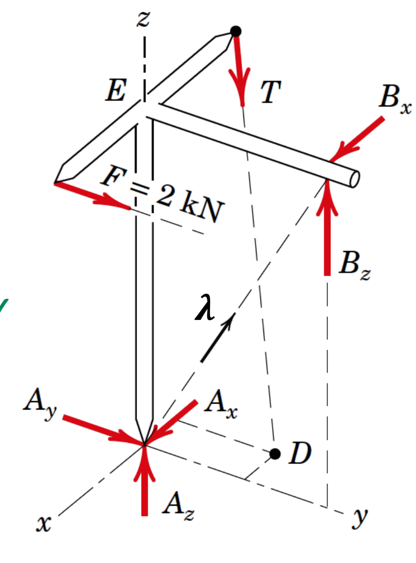


Finding tension T:

$\Sigma \mathbf{M}_{AB} = \mathbf{0} \rightarrow T$ ✓

vs

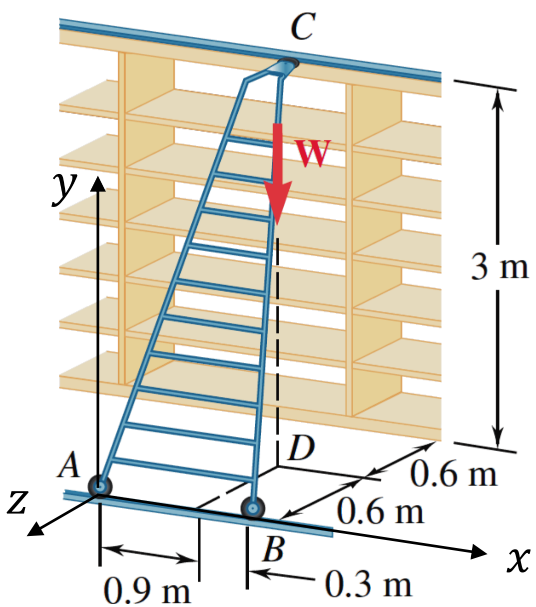
$\Sigma \mathbf{M}_{A} = \mathbf{0} \rightarrow B_x, B_z, T$ ✓



In this way, we can eliminate the reactions at two points A and B by writing $\Sigma \mathbf{M}_{AB} = \mathbf{0}$, which involves the computation of the moments of the forces about an axis AB joining points A and B.

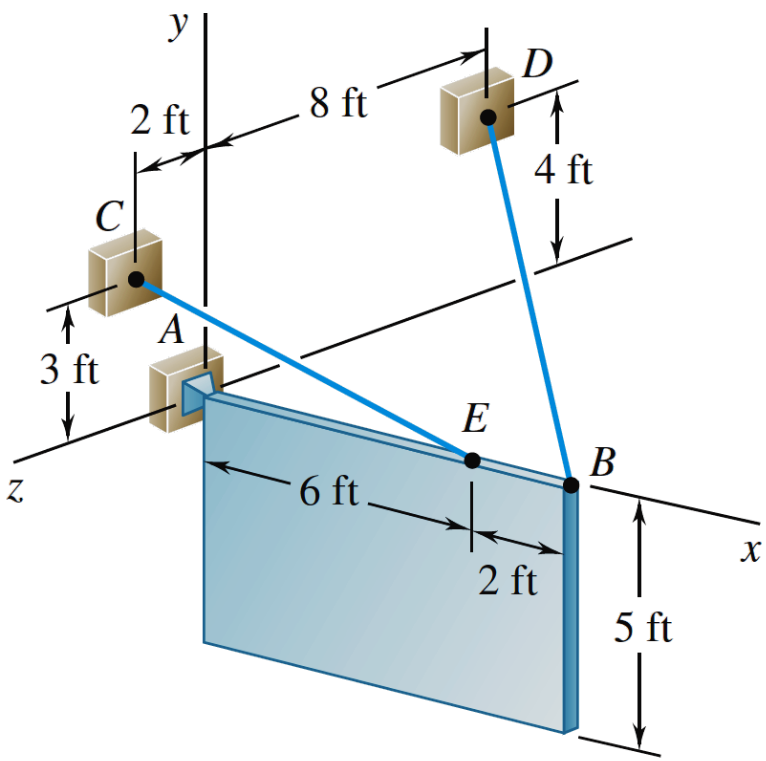
Sample Problem 4.7

A 20-kg ladder used to reach high shelves in a storeroom is supported by two flanged wheels A and B mounted on a rail and by a flangeless wheel C resting against a rail fixed to the wall. (The ladder itself is symmetric, with wheel C located on the plane of symmetry.) An 80-kg man stands on the ladder and leans to the right. The line of action of the combined weight **W** of the man and ladder intersects the floor at point D. Determine the reactions at A, B, and C.



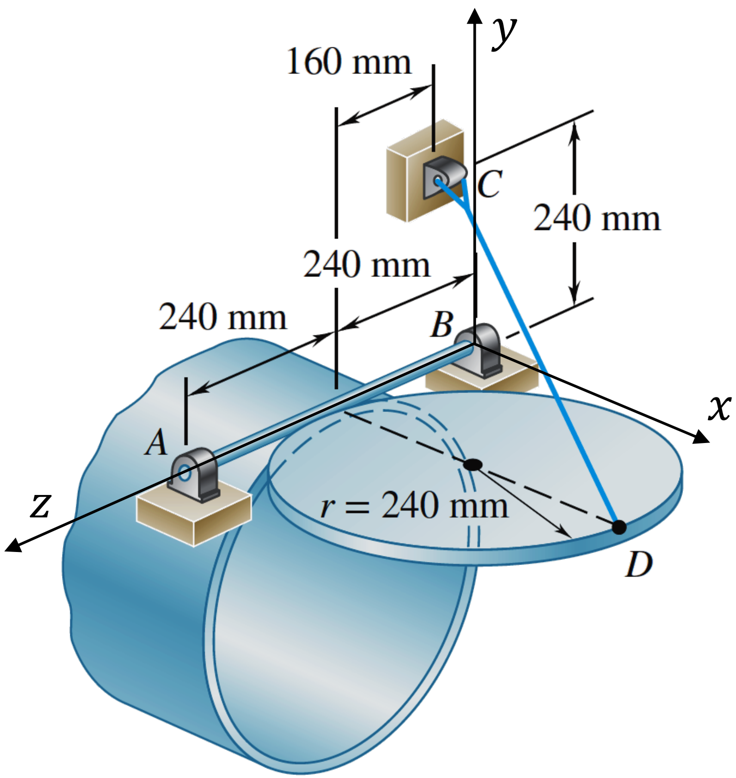
Sample Problem 4.8

A 5 × 8-ft sign of uniform density weighs 270 lb and is supported by a ball-and-socket joint at A and by two cables. Determine the tension in each cable and the reaction at A.



Sample Problem 4.9

A uniform pipe cover of radius $r = 240$ mm and mass $m = 30$ kg is held in a horizontal position by the cable CD. Assuming that only the bearing at B does not exert any axial thrust, determine the tension in the cable and the reactions at A and B.

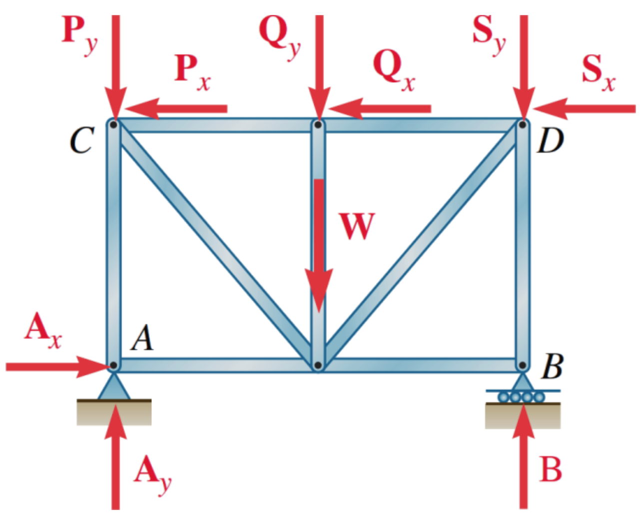


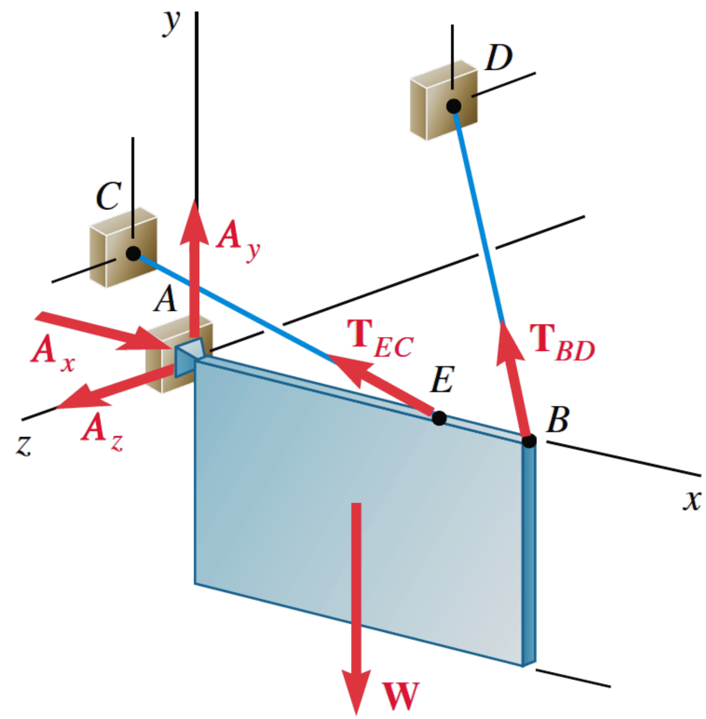
Statical Determinacy and Constraints

Statical Determinacy

A rigid body which has **the minimum number of supports/constraints** necessary to maintain an equilibrium configuration under given loads is called **statically determinate**. For such bodies,

- The equilibrium equations are **sufficient** to determine all the unknown reactions (up to three unknowns in 2D or six unknowns in 3D).





Statical Determinacy

A rigid body which has **more supports/constraints than are necessary** to maintain an equilibrium configuration under given loads is called **statically indeterminate**. For such bodies

- The equilibrium equations are **insufficient** to determine all the unknown reactions (e.g., **more than** three unknowns in 2D or six unknowns in 3D).
- You may be able to calculate **a few of reactions**. The remaining reactions can be determined by additional equations obtained from the deformation conditions at the points of support (MEC 363 [Mechanics of Solids]).