Ch9: Distributed Forces: Moments of Inertia

Contents:

Moments of Inertia of Areas (or Second Moment of Areas)

Parallel-Axis Theorem and Composite Areas

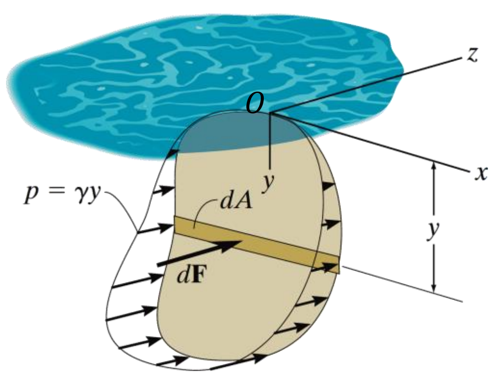
Mass Moments of Inertia

Moments of Inertia of Areas (or Second Moment of Areas)

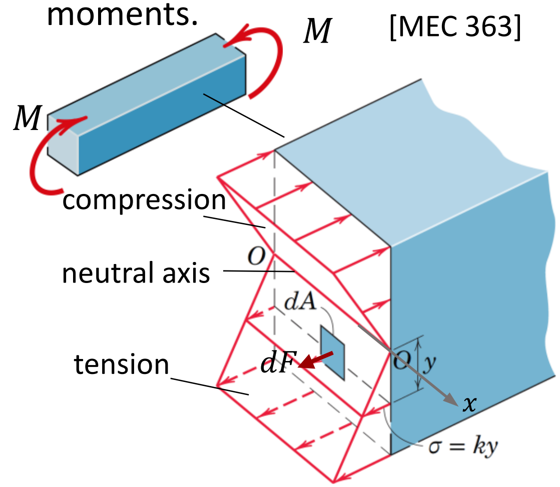
Motivation

In these examples of distributed forces, the force element dF is proportional to the element of area dA on which dF act and, at the same time, vary linearly with the distance from dA to a given axis.

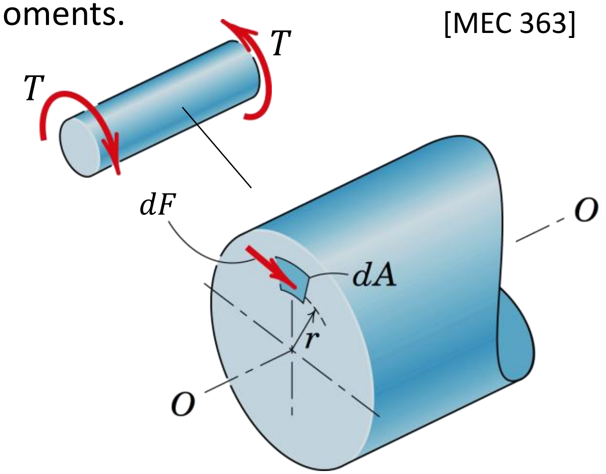
(1) A vertical surface submerged in a fluid. [MEC 364]



(2) A beam subjected to bending moments. [MEC 363]



(3) A circular shaft subjected to torsional moments. [MEC 363]



$dF = pdA = \rho g y dA \rightarrow F = \rho g \int y dA$

$dM_x = ydF = \rho g y^2 dA \rightarrow M_x = \rho g \int y^2 dA$

$dF = \sigma dA = kydA$

$dM_x = ydF = ky^2 dA$

$dF = \tau dA = krdA$

$dM_O = rdF = kr^2 dA$

∫(distance) d(area): First Moment of an Area

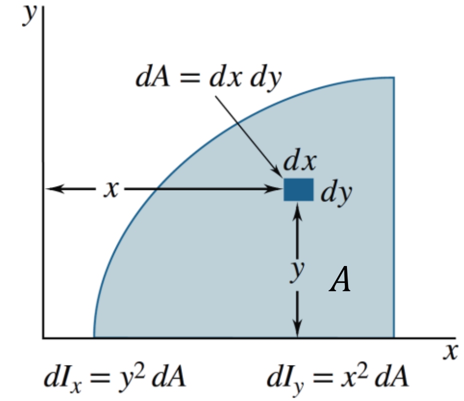
∫(distance)² d(area): Second Moment of an Area (or Moment of Inertia of an Area)

(Purely mathematical properties of the area)

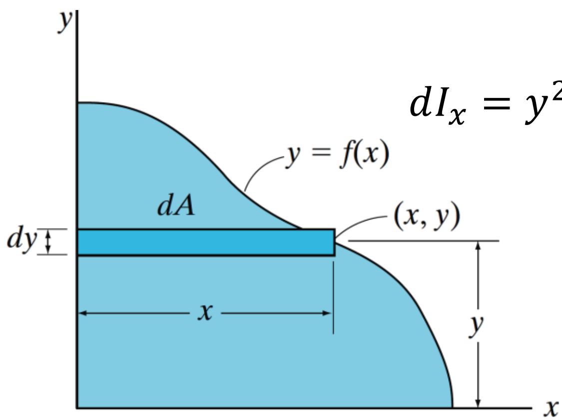
Moment of Inertia of an Area
(or Second Moment of an Area)

By definition, the **moments of inertia** of a differential area dA about the x and y axes are $dI_x = y^2 dA$ and $dI_y = x^2 dA$, respectively. For the entire area A, the moments of inertia are determined by integration:

$$I_x = \int dI_x = \int y^2 dA \qquad I_y = \int dI_y = \int x^2 dA$$



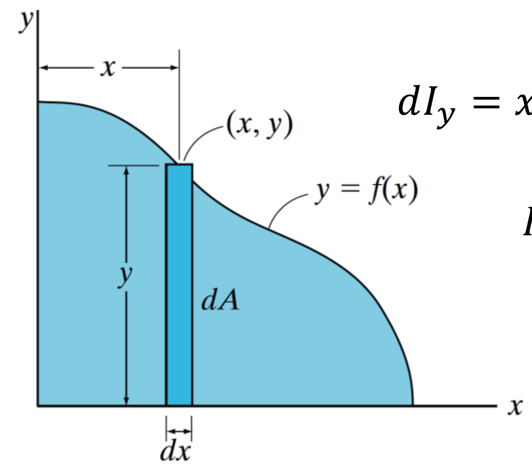
To compute I_x (or I_y) using a single integration, choose a thin strip parallel to the x axis (or y axis), so that all points of the strip are at the same distance from the axis.



$$dI_x = y^2 dA = y^2 x dy$$

$$I_x = \int dI_x$$

❶ and



$$dI_y = x^2 dA = x^2 y dx$$

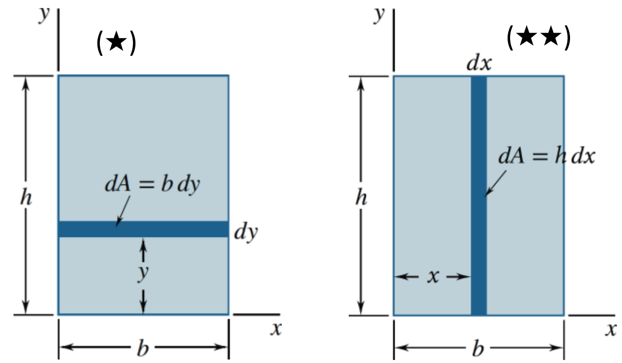
$$I_y = \int dI_y$$

Moment of Inertia of an Area
(or Second Moment of an Area) (cont.)

Example: Moment of Inertia of a Rectangular Area.

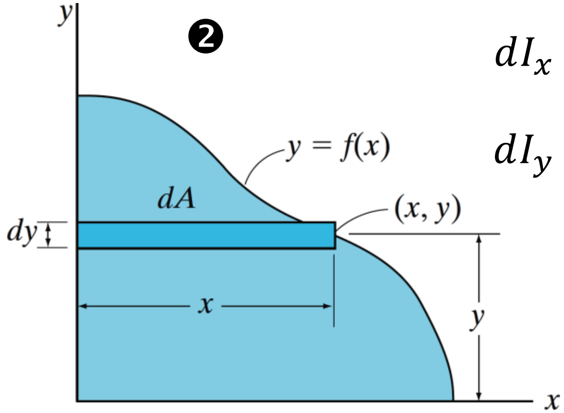
(★)
$$dI_x = y^2 dA = y^2 b dy \longrightarrow I_x = \int_0^h b y^2 dy = \frac{1}{3} b h^3$$

(★★)
$$dI_y = x^2 dA = x^2 h dx \longrightarrow I_y = \int_0^b x^2 h dx = \frac{1}{3} h b^3$$



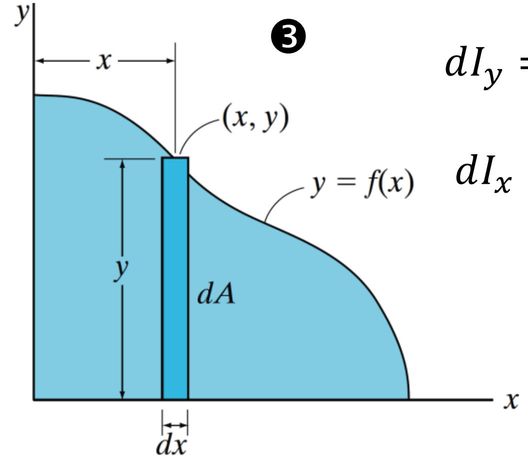
$\Downarrow$

Note: By using I_x, I_y computed for a rectangular area, we can now use <u>only one rectangular element</u> dA (vertical or horizontal) to compute both moments of inertia I_x and I_y for a given area (when one side of the element dA is on the x or y axis):



$$dI_x = y^2 dA = y^2 x dy$$
$$dI_y = \frac{1}{3} x^3 dy$$
$$I_x = \int dI_x$$
$$I_y = \int dI_y$$

or

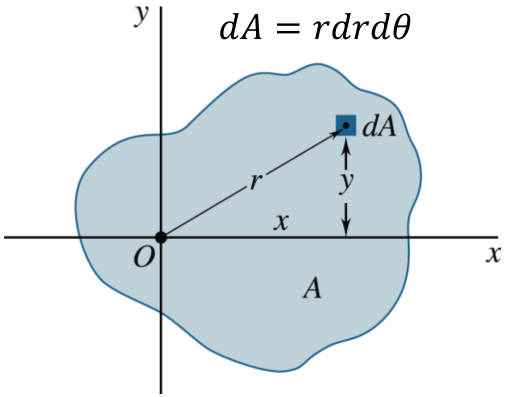


$$dI_y = x^2 dA = x^2 y dx$$
$$dI_x = \frac{1}{3} y^3 dx$$
$$I_x = \int dI_x$$
$$I_y = \int dI_y$$

Polar Moment of Inertia of an Area

By formulating the moments of inertia of dA about the "**pole**" O or z axis, we can define the **polar moment of inertia** as

$$J_O = \int r^2 dA$$



r: the perpendicular distance from O (z axis) to the element dA

$$r^2 = x^2 + y^2$$

$$J_O = \int r^2 dA = \int (x^2 + y^2) dA = \int x^2 dA + \int y^2 dA$$

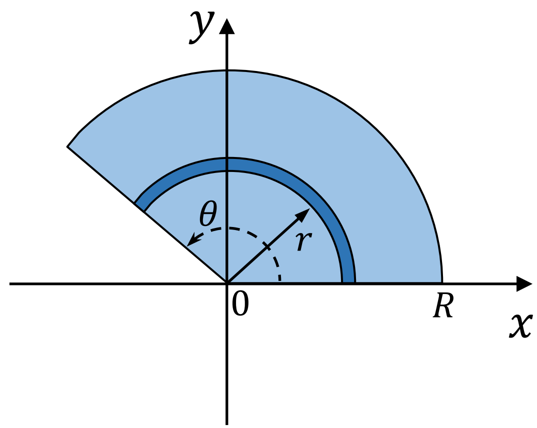
$$J_O = I_y + I_x$$

Note: I_x, I_y, and J_O are always positive.

Note: The unit of I_x, I_y, and J_O involve length to the power of 4, e.g., m^4, mm^4, or ft^4, in.^{4}.

Finding Polar Moment of Inertia of an Area

- If the given area has circular symmetry, it is possible to express dA as a function of r and to compute J_O with a **single integration**.



$$dJ_O = r^2 dA = r^2 (r\theta dr)$$

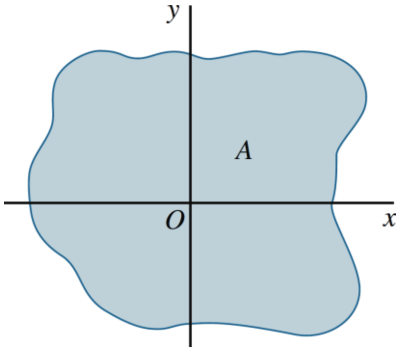
$$J_O = \int dJ_O = \theta \int_0^R r^3 dr$$

- If the area lacks circular symmetry, it is usually easier first to calculate I_x and I_y and then to determine J_O from $J_O = I_y + I_x$.

Radius of Gyration of an Area

Consider an area A that has a moments of inertia I_x, I_y, and J_O. The **radius of gyration** k_x, k_y, k_O of the area about x axis, y axis, and O (z axis) are defined as:

$$k_x = \sqrt{\frac{I_x}{A}} \qquad k_y = \sqrt{\frac{I_y}{A}} \qquad k_O = \sqrt{\frac{J_O}{A}}$$

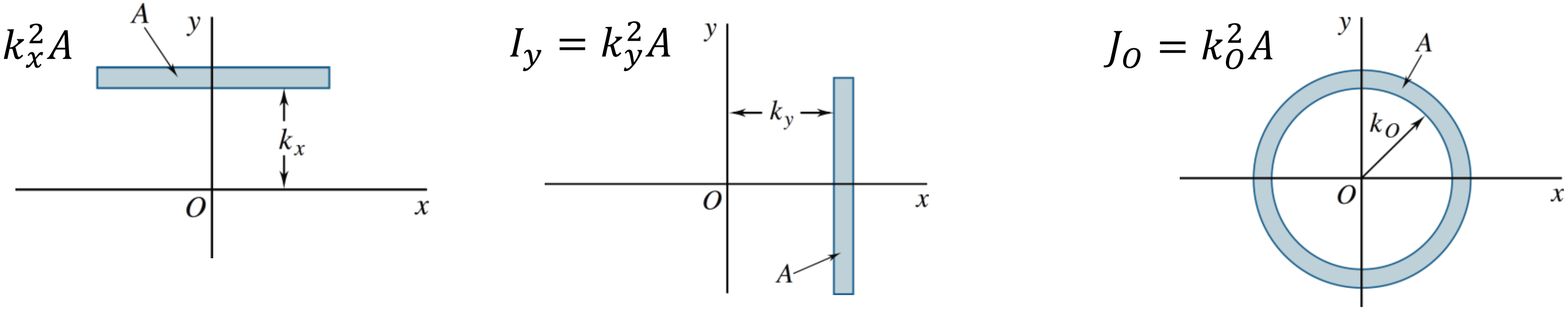


Note: From $J_O = I_x + I_y$, we have: $k_O^2 = k_x^2 + k_y^2$

Note: Radius of gyration has units of **length**, e.g., m, mm, or ft, in..

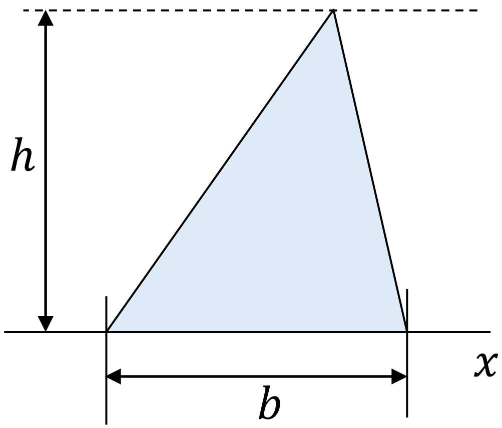
Mathematical Interpretation of Radius of Gyration: Imagine that the area A is concentrated into a thin strip parallel to the x axis. To have the same moment of inertia with respect to the x axis, the strip should be placed at a distance k_x from the x axis. k_y and k_O are defined in a similar way.

$I_x = k_x^2 A$ $\qquad$ $I_y = k_y^2 A$ $\qquad$ $J_O = k_O^2 A$



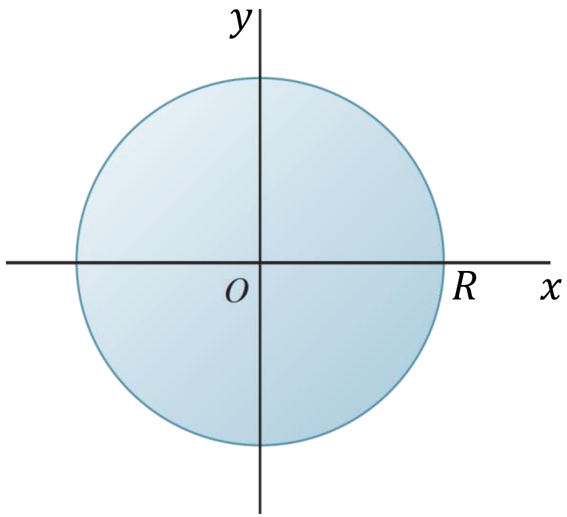
Sample Problem 9.1

Determine the moment of inertia of a triangle with respect to its base.



Sample Problem 9.2

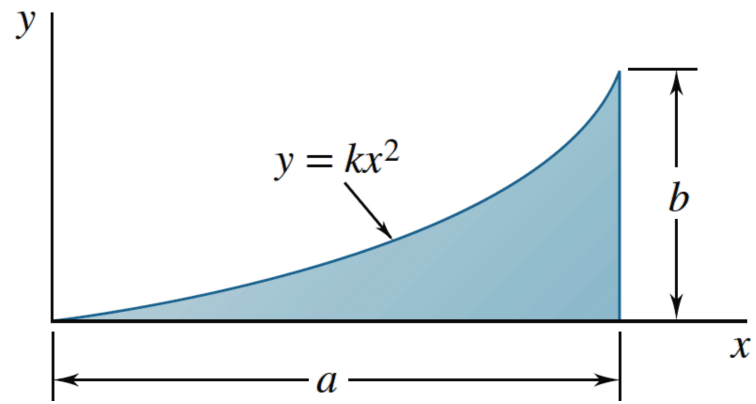
(a) Determine the centroidal polar moment of inertia of a circular area by direct integration. (b) Using the result of part a, determine the moment of inertia of a circular area with respect to a diameter.



Sample Problem 9.3

(a) Determine the moment of inertia of the shaded region shown with respect to each of the coordinate axes. (b) Using the results of part a, determine the radius of gyration of the shaded area with respect to each of the coordinate axes.

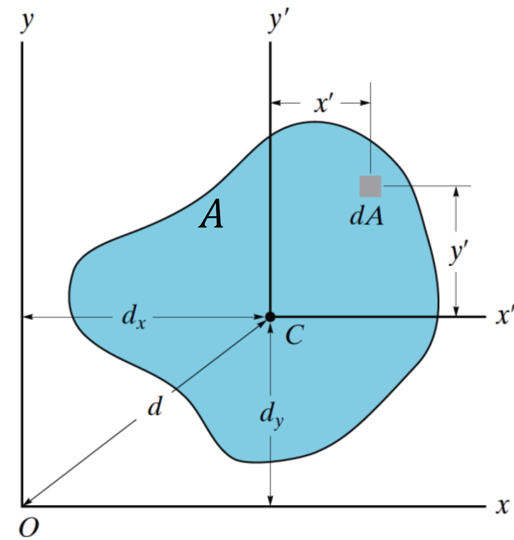
$$A = \frac{ab}{3}$$



Parallel-Axis Theorem and Composite Areas

Parallel-Axis Theorem for an Area

Let C be the centroid of the area A, x' and y' axes be centroidal axes, $\bar{I}_{x'}$ and $\bar{I}_{y'}$ be the moment of inertia of the area about the centroidal axes, and x and y axes be two arbitrary axes parallel to centroidal axes x' and y' at distances d_y and d_x, respectively.



$$I_x = \int (y' + d_y)^2 dA = \int y'^2 dA + 2d_y \int y' dA + d_y^2 \int dA$$

0 — The centroid C is located on the axis x'

$$I_x = \bar{I}_{x'} + Ad_y^2$$

Similarly, $$I_y = \bar{I}_{y'} + Ad_x^2$$

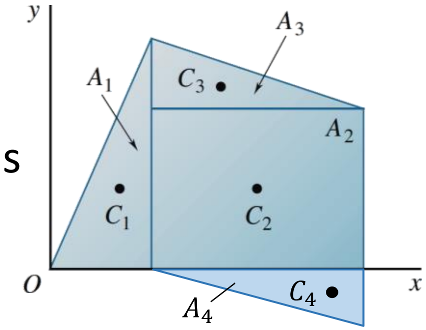
$$\begin{cases} \bar{J}_C = \bar{I}_{x'} + \bar{I}_{y'} \\ d^2 = d_x^2 + d_y^2 \end{cases} \rightarrow \quad J_O = I_x + I_y = \bar{J}_C + Ad^2$$

$$\rightarrow \begin{cases} k_x^2 = \bar{k}_{x'}^2 + d_y^2 \\ k_y^2 = \bar{k}_{y'}^2 + d_x^2 \\ k_O^2 = \bar{k}_C^2 + d^2 \end{cases}$$

★ The moment of inertia for an area about <u>an arbitrary axis</u> is equal to its moment of inertia about a <u>parallel axis passing through the area's **centroid**</u> plus the product of the area and the square of the perpendicular distance between the axes.

Moments of Inertia of Composite Areas

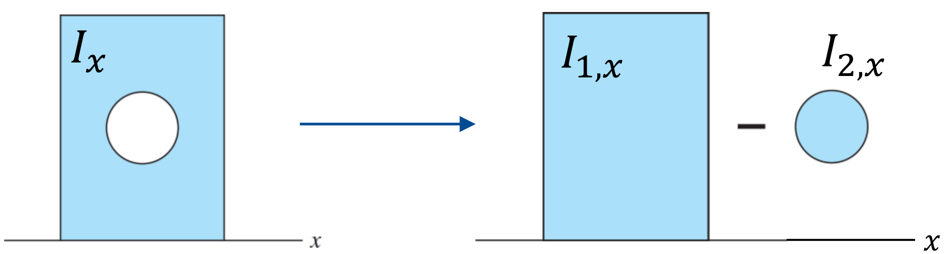
Consider a composite area A can be broken down into a sum of simple areas $A_1, A_2, A_3, \ldots$. The moment of inertia of A about a given axis can be obtained by algebraically adding the moments of inertia of the areas $A_1, A_2, A_3, \ldots$ with respect to the same axis.



$$I_x = I_{1,x} + I_{2,x} + I_{3,x} + I_{4,x} + \cdots$$

Note: It is usually required to first determine the **perpendicular distance** from the **centroid** of each component to the reference axis, then, use the **parallel-axis theorem** to determine the moment of inertia of the components about the same reference axis.

Note: If a composite part has an empty region (hole), its moment of inertia is found by subtracting the moment of inertia of this region from the moment of inertia of the entire part including the region.

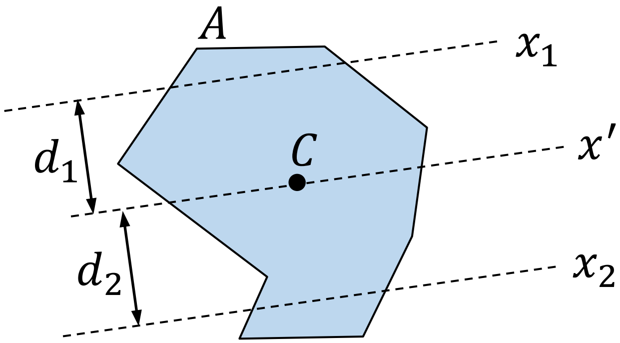


$$I_x = I_{1,x} - I_{2,x}$$

Remarks

Note: The **radius of gyration of a composite area** is not equal to the sum of the radii of gyration of the component areas. In order to determine the radius of gyration of a composite area, you must first compute the moment of inertia I of the composite area A, and then $k = \sqrt{I/A}$.

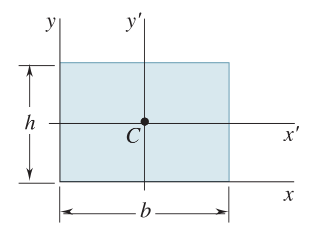
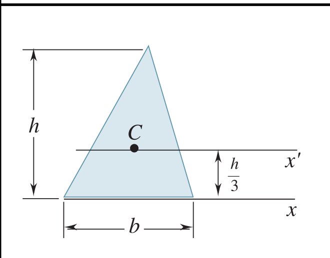
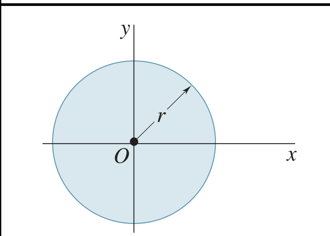
Note: To compute the moment of inertia of an area with respect to a **noncentroidal axis** (I_{x_2}) when the moment of inertia of the area is **known with respect to another parallel noncentroidal axis** (I_{x_1}), it is necessary to **first** compute the moment of inertia of the area with respect to the **centroidal axis parallel to the two given axes** ($\bar{I}_{x'}$) using the parallel-axis theorem, and then use the parallel-axis theorem again to find I_{x_2}.

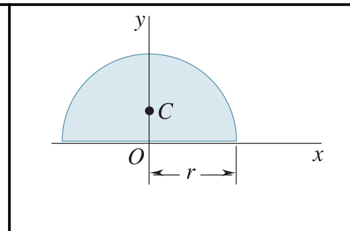
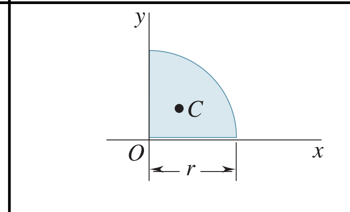
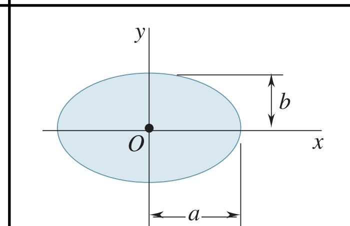


$$I_{x_1} = \bar{I}_{x'} + Ad_1^2 \longrightarrow \bar{I}_{x'} \checkmark$$

$$I_{x_2} = \bar{I}_{x'} + Ad_2^2 \longrightarrow I_{x_2} \checkmark$$

Moments of Inertia of Common Shapes

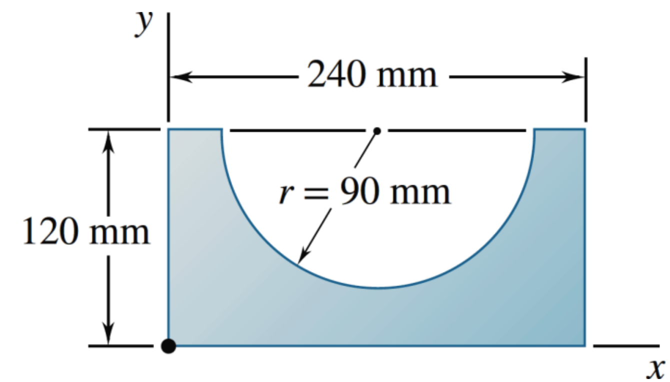
Rectangle		$\bar{I}_{x'} = \frac{1}{12} bh^3$ $\bar{I}_{y'} = \frac{1}{12} b^3 h$ $I_x = \frac{1}{3} bh^3$ $I_y = \frac{1}{3} b^3 h$ $J_C = \frac{1}{12} bh(b^2 + h^2)$
Triangle		$\bar{I}_{x'} = \frac{1}{36} bh^3$ $I_x = \frac{1}{12} bh^3$
Circle		$\bar{I}_x = \bar{I}_y = \frac{1}{4} \pi r^4$ $J_O = \frac{1}{2} \pi r^4$

Semicircle		$I_x = I_y = \frac{1}{8} \pi r^4$ $J_O = \frac{1}{4} \pi r^4$
Quarter circle		$I_x = I_y = \frac{1}{16} \pi r^4$ $J_O = \frac{1}{8} \pi r^4$
Ellipse		$\bar{I}_x = \frac{1}{4} \pi ab^3$ $\bar{I}_y = \frac{1}{4} \pi a^3 b$ $J_O = \frac{1}{4} \pi ab(a^2 + b^2)$

★ The moments of inertia of a semicircle/semiellipse and a quarter circle/ellipse can be determined by dividing the moment of inertia of a circle/ellipse by 2 and 4, respectively. Note that the moments of inertia obtained in this manner are **about the axes of symmetry** of the circle/ellipse. To obtain the centroidal moments of inertia of these shapes, use the **parallel-axis theorem**.

Sample Problem 9.5

Determine the moment of inertia of the shaded area with respect to the x axis.

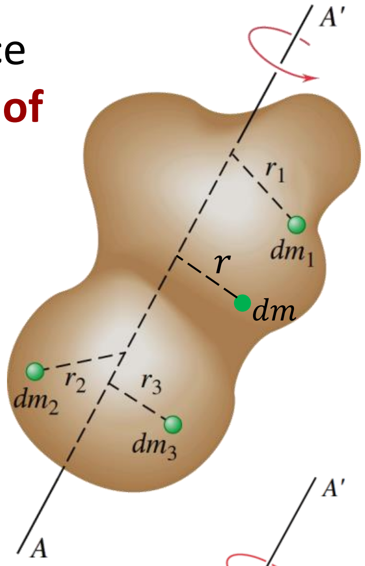


Mass Moments of Inertia

Mass Moments of Inertia

The mass moment of inertia of a body is a measure of the body's resistance to angular acceleration (rotational motion) [MEC 262]. The **mass moment of inertia** of a body about an axis AA' is defined as

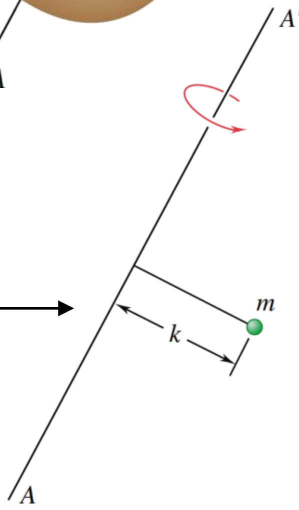
$$I = \int r^2 dm$$



Radius of gyration of a mass about axis AA' is defined as

$$I = k^2 m \quad \longrightarrow \quad k = \sqrt{\frac{I}{m}}$$

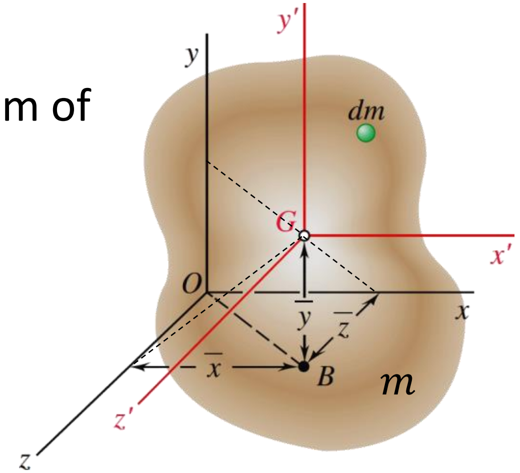
Note: k represents the distance at which the entire mass of the body should be concentrated if its moment of inertia with respect to AA' is to remain unchanged.



Note: Unit for the mass moment of inertia I is kg·m^2, slug.ft^2, or lb·ft·s^2 and unit for the radius of gyration k is m or ft.

Parallel-Axis Theorem for a Mass

Let G be the center of gravity/mass of the mass m, $G_{x'y'z'}$ be a coordinate system whose origin is at G, O_{xyz} be a coordinate system of parallel axes whose origin is at the arbitrary point O.



$$I_x = \int r_x^2 dm = \int (y^2 + z^2) dm = \int [(y' + \bar{y})^2 + (z' + \bar{z})^2] dm$$
$$= \int (y'^2 + z'^2) dm + 2\bar{y} \int y' dm + 2\bar{z} \int z' dm + (\bar{y}^2 + \bar{z}^2) \int dm$$

0 0

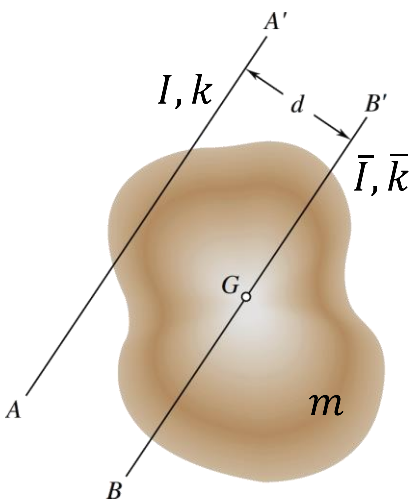
G is located on the axes y' and z'

$$I_x = \bar{I}_x + m(\bar{y}^2 + \bar{z}^2)$$

Similarly,
$$I_y = \bar{I}_y + m(\bar{x}^2 + \bar{z}^2)$$
$$I_z = \bar{I}_z + m(\bar{x}^2 + \bar{y}^2)$$

A general relation

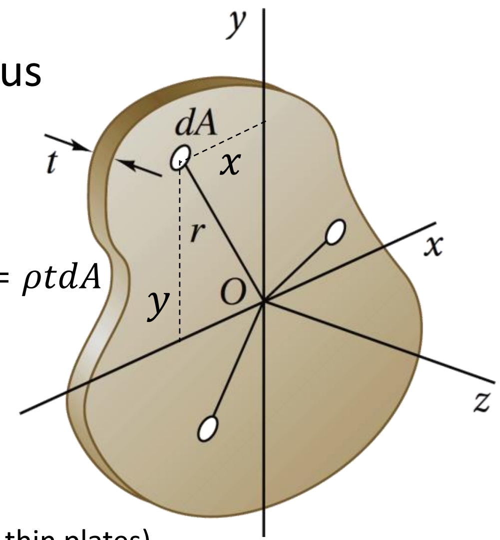
$$I = \bar{I} + md^2$$



For radius of gyration: $k^2 = \bar{k}^2 + d^2$

Mass Moments of Inertia of Thin Plates

Consider a thin plate of uniform thickness t, made of a homogeneous material of density ρ.



$$dm = \rho dV = \rho t dA$$

$$I_{x,\text{mass}} = \int y^2 dm = \rho t \int y^2 dA \rightarrow I_{x,\text{mass}} = \rho t I_{x,\text{area}}$$

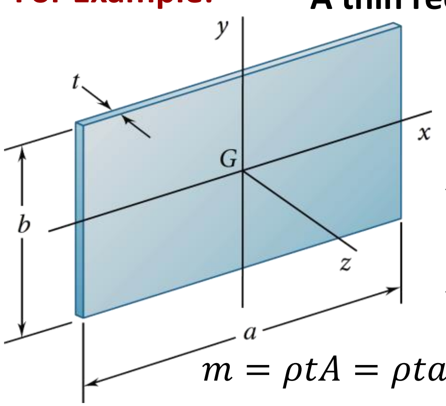
$$I_{y,\text{mass}} = \int x^2 dm = \rho t \int x^2 dA \rightarrow I_{y,\text{mass}} = \rho t I_{y,\text{area}}$$

$$I_{z,\text{mass}} = \int r^2 dm = \rho t \int r^2 dA \rightarrow I_{z,\text{mass}} = \rho t J_{O,\text{area}}$$

Since $J_{O,\text{area}} = I_{x,\text{area}} + I_{y,\text{area}} \rightarrow I_{z,\text{mass}} = I_{x,\text{mass}} + I_{y,\text{mass}}$ (only for thin plates)

For Example:

A thin rectangular plate:



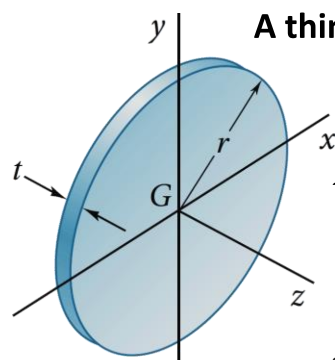
$$m = \rho t A = \rho t a b$$

$$I_{x,\text{mass}} = \rho t \left(\frac{1}{12} a b^3\right) = \frac{1}{12} m b^2$$

$$I_{y,\text{mass}} = \rho t \left(\frac{1}{12} b a^3\right) = \frac{1}{12} m a^2$$

$$I_{z,\text{mass}} = \frac{1}{12} m (a^2 + b^2)$$

A thin circular plate:



$$m = \rho t A = \rho t \pi r^2$$

$$I_{x,\text{mass}} = \rho t \left(\frac{1}{4} \pi r^4\right) = \frac{1}{4} m r^2$$

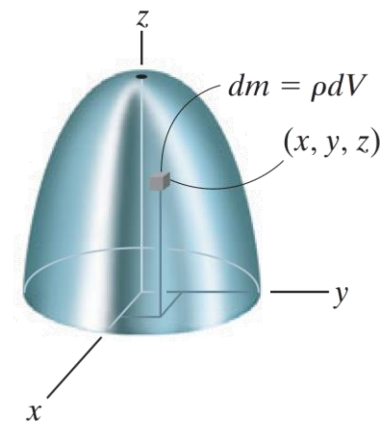
$$I_{y,\text{mass}} = \frac{1}{4} m r^2$$

$$I_{z,\text{mass}} = \frac{1}{2} m r^2$$

Determining Mass Moment of Inertia

In general, if the body is made of a homogeneous material with a density ρ,

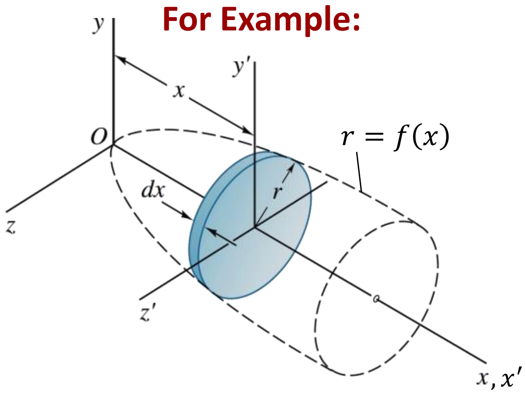
$$I = \int r^2 dm = \rho \int r^2 dV = \rho \int r^2 dx dy dz$$



Thus, it is generally necessary to perform a **triple**, or at least a **double**, **integration** which is purely a function of geometry.

However, it is usually possible to determine the body's mass moment of inertia with a **single integration**. In this cases, we first divide the body into a series of thin, parallel slabs. Then, compute the moment of inertia of the slab with respect to the given axis (use the parallel-axis theorem if necessary) and finally, **integrate** the resulting expression.

For Example:



$$dm = \rho dV = \rho \pi r^2 dx$$

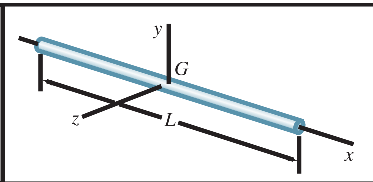
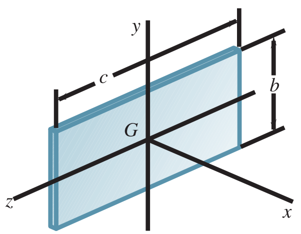
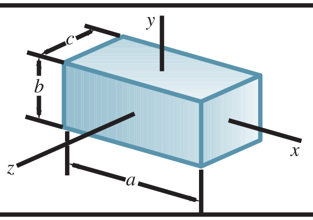
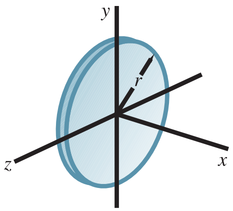
$$dI_x = dI_{x'} = \frac{1}{2} r^2 dm$$ ($dI_{x'}, dI_{y'}, dI_{z'}$ from previous slide.)

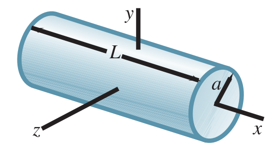
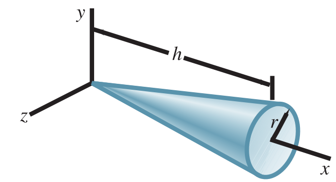
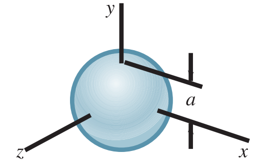
$$dI_y = dI_{y'} + x^2 dm = \left(\frac{1}{4} r^2 + x^2\right) dm$$

$$dI_z = dI_{z'} + x^2 dm = \left(\frac{1}{4} r^2 + x^2\right) dm$$

(You can use a similar method for **rectangular** and **triangular** slabs.)

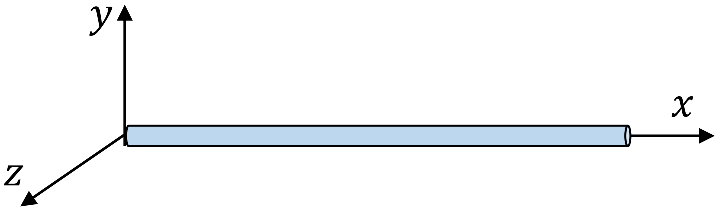
Mass Moments of Inertia of Common Shapes

Slender rod		$I_y = I_z = \frac{1}{12} mL^2$
Thin rectangular plate		$I_x = \frac{1}{12} m(b^2 + c^2)$ $I_y = \frac{1}{12} mc^2$ $I_z = \frac{1}{12} mb^2$
Rectangular prism		$I_x = \frac{1}{12} m(b^2 + c^2)$ $I_y = \frac{1}{12} m(c^2 + a^2)$ $I_z = \frac{1}{12} m(a^2 + b^2)$
Thin disk		$I_x = \frac{1}{2} mr^2$ $I_y = I_z = \frac{1}{4} mr^2$

Circular cylinder		$I_x = \frac{1}{2} ma^2$ $I_y = I_z = \frac{1}{12} m(3a^2 + L^2)$
Circular cone		$I_x = \frac{3}{10} ma^2$ $I_y = I_z = \frac{3}{5} m\left(\frac{1}{4}a^2 + h^2\right)$
Sphere		$I_x = I_y = I_z = \frac{2}{5} ma^2$

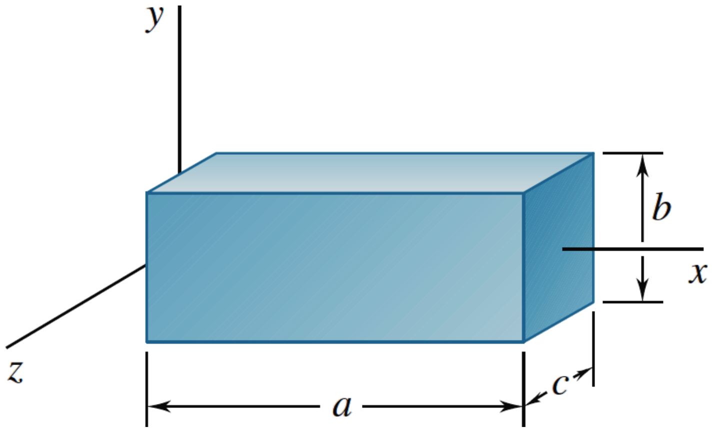
Sample Problem 9.9

Determine the moment of inertia of a slender rod of length L and mass m with respect to an axis that is perpendicular to the rod and passes through one end.



Sample Problem 9.10

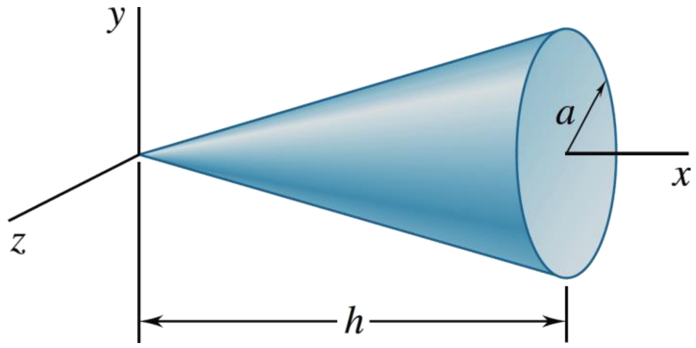
For the homogeneous rectangular prism shown, determine the moment of inertia with respect to the z-axis.



Sample Problem 9.11

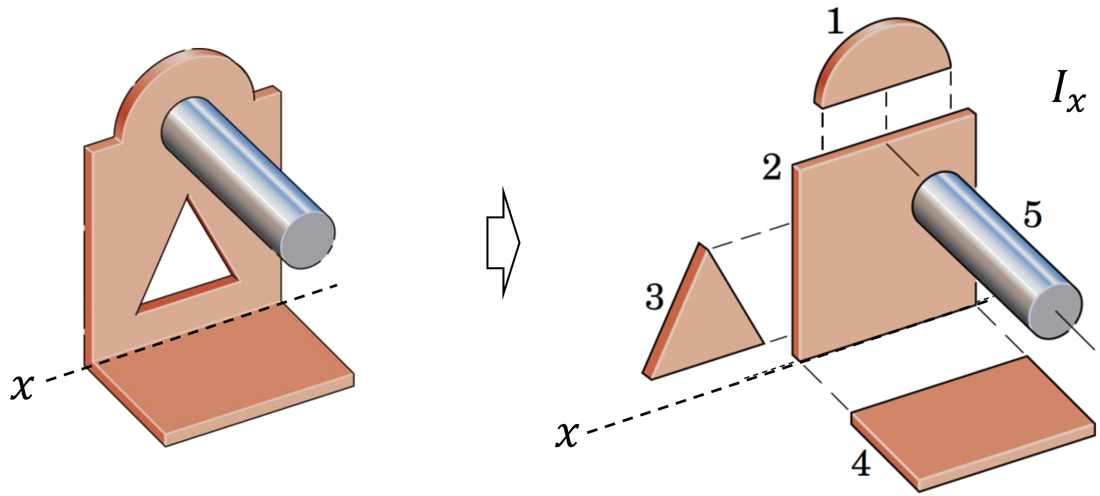
Determine the moment of inertia of a right circular cone with respect to (a) its longitudinal axis, (b) an axis through the apex of the cone and perpendicular to its longitudinal axis, (c) an axis through the centroid of the cone and perpendicular to its longitudinal axis.

$$\bar{x} = \frac{3}{4}h$$



Moments of Inertia of Composite Bodies

In many instances, we can **divide** a body **into the common shapes** given in the tables. We can obtain the moment of inertia of the body with respect to a given axis by first computing the moments of inertia of its component parts about the given axis (you may need to use the parallel-axis theorem) and then algebraically adding them together.



$$I_x = I_{1x} + I_{2x} - I_{3x} + I_{4x} + I_{5x}$$

Note: If a composite part has an empty region (hole), its moment of inertia is found by subtracting the moment of inertia of this region from the moment of inertia of the entire part including the region.

Note: As was the case for areas, the radius of gyration of a composite body cannot be obtained by adding the radii of gyration of its component parts. You must first compute the moment of inertia of the composite area, and then $k = \sqrt{I/m}$.

Sample Problem 9.12

A steel forging consists of a 6×2×2-in. rectangular prism and two cylinders with a diameter of 2 in. and length of 3 in. as shown. Determine the moments of inertia of the forging with respect to the coordinate axes. The specific weight of steel is 490 lb/ft3.