Ch2: Laplace Transform Review

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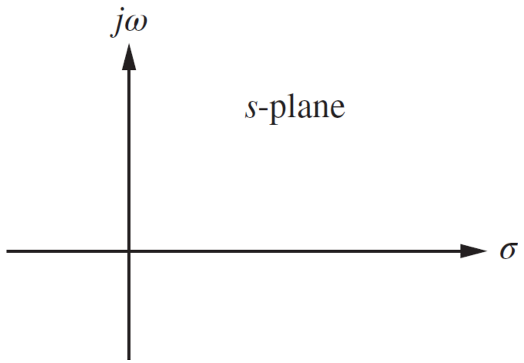
Laplace Transform

Laplace Transform

The definition of the unilateral (or one-sided) **Laplace Transform** is:

$$\mathcal{L}[f(t)] = F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt$$

where $s = \sigma + j\omega$ is a **complex variable** (with real numbers σ and ω) and 0^- is a value just before $t = 0$ (which is applicable for discontinuous functions like impulse function or discontinuous initial conditions of differential equations at $t = 0$).



Note: The Laplace Transform exists if there exists a real number σ_1 such that:

$$\lim_{t\to\infty} |f(t)e^{-\sigma_1 t}| = 0$$

Example

Find the Laplace transform of $f(t) = Ae^{-at}$ $(t \geq 0)$.

Answer: $F(s) = \dfrac{A}{s+a}$

Inverse Laplace Transform

Finding $f(t)$ from $F(s)$:

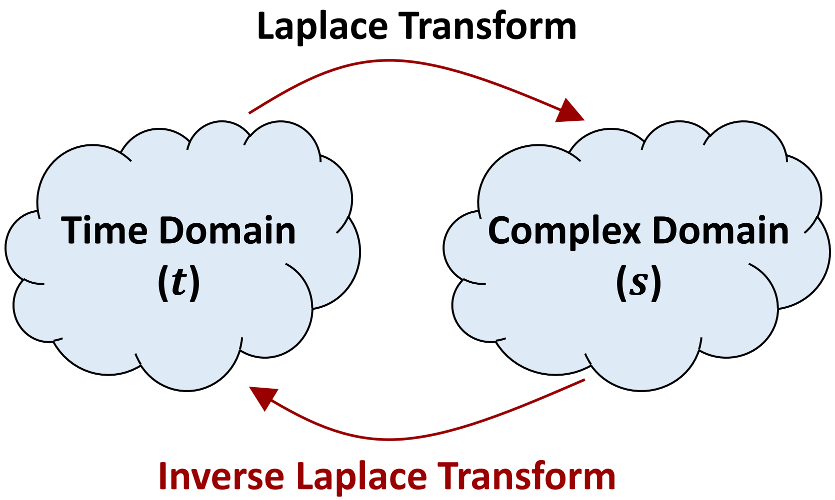
$$\mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds = f(t) \quad (t \geq 0)$$

where σ, the abscissa of convergence, is a real constant and is chosen larger than the real parts of all singular points of $F(s)$. Thus, the path of integration is parallel to the $j\omega$ axis and is displaced by the amount σ from it. This path of integration is to the right of all singular points.

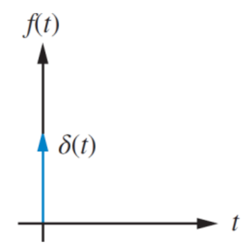
Evaluating the inversion integral appears complicated. In practice, we frequently use the **Laplace Transform Theorems** and **Partial-Fraction Expansion Method** for transforming between $f(t)$ and $F(s)$.

Laplace Transform Pairs

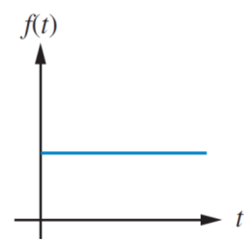
	$f(t)$, $t > 0$	$F(s)$
(impulse)	$\delta(t)$	1
(step)	$u(t)$	$\dfrac{1}{s}$
	t^n	$\dfrac{n!}{s^{n+1}}$
	e^{-at}	$\dfrac{1}{s+a}$
	$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$
	$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$



$$\delta(t) = \infty \text{ for } 0- < t < 0+$$
$$= 0 \text{ elsewhere}$$
$$\int_{0-}^{0+} \delta(t)dt = 1$$



$$u(t) = 1 \text{ for } t > 0$$
$$= 0 \text{ for } t < 0$$



Laplace Transform Theorems

No.	Theorem	Name
1.	$\mathcal{L}[k_1 f_1(t) + k_2 f_2(t)] = k_1 F_1(s) + k_2 F_2(s)$	Linearity theorem
2.	$\mathcal{L}[e^{-at} f(t)] = F(s+a)$	Frequency shift theorem
3.	$\mathcal{L}[f(t-T)] = e^{-sT} F(s)$	Time shift theorem
4.	$\mathcal{L}[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$	Scaling theorem
5.	$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0^-)$	Differentiation theorem
6.	$\mathcal{L}\left[\frac{d^2 f(t)}{dt^2}\right] = s^2 F(s) - sf(0^-) - f'(0^-)$	Differentiation theorem
7.	$\mathcal{L}\left[\frac{d^n f(t)}{dt^n}\right] = s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{k-1}(0^-)$	Differentiation theorem
8.	$\mathcal{L}\left[\int_{0^-}^{t} f(\tau) d\tau\right] = \frac{F(s)}{s}$	Integration theorem

Laplace Transform Theorems

No.	Theorem	Name
9.	$\mathcal{L}[tf(t)] = -\dfrac{dF(s)}{ds}$	Multiplication by time
10.	$\mathcal{L}[t^n f(t)] = (-1)^n \dfrac{d^n F(s)}{ds^n}$, $(n = 1,2, \ldots)$	Multiplication by time
11.	$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$	Final value theorem [1]
12.	$\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} s\,F(s)$	Initial value theorem [2]
13.	$\mathcal{L}^{-1}[F_1(s)F_2(s)] = f_1(t) * f_2(t)$	Convolution Integral [3]

[1] For this theorem to yield correct finite results, all **roots** of the denominator of $F(s)$ must have **negative real parts**, and no more than one can be at the origin.

[2] For this theorem to be valid, $f(t)$ must be **continuous** or have a step discontinuity at $t = 0$ (that is, no impulses or their derivatives at $t = 0$).

[3] $f_1(t) * f_2(t) = \int_0^t f_1(t-\tau) f_2(\tau) d\tau = \int_0^t f_1(\tau) f_2(t-\tau) d\tau$ and $f_1(t)$ and $f_2(t)$ are 0 for $t < 0$.

Example

$f(t) = 1 + 2\sin\omega t \qquad \mathcal{L}\{f(t)\} = ?$

$f(t) = A\sin(t - t_d) \qquad \mathcal{L}\{f(t)\} = ?$

$f(t) = Ae^{-at}\sin\omega t \qquad \mathcal{L}\{f(t)\} = ?$

$F(s) = \dfrac{1}{(s+3)^2} \qquad \mathcal{L}^{-1}\{F(s)\} = ?$

$F(s) = \dfrac{1}{s^2(s-a)} \qquad \mathcal{L}^{-1}\{F(s)\} = ?$

Answers:

$$F(s) = \frac{s^2 + 2\omega s + \omega^2}{s^3 + \omega^2 s}$$

$$F(s) = \frac{A}{s^2+1}e^{-st_d}$$

$$F(s) = \frac{A\omega}{(s+a)^2 + \omega^2}$$

$$f(t) = e^{-3t}t$$

$$f(t) = \frac{1}{a^2}(e^{at} - at - 1)$$

Partial-Fraction Expansion

Partial-Fraction Expansion

To find the **inverse Laplace transform** of a complicated function $F(s) = N(s)/D(s)$, we can **convert** the function to a sum of **simpler terms** for which we know the Laplace transform of each term using the Tables and Theorems.

If the order of $N(s)$ is less than the order of $D(s)$, then a Partial-Fraction Expansion can be made. If the order of $N(s)$ is greater than or equal to the order of $D(s)$, then first $N(s)$ must be divided by $D(s)$ successively until the result has a remainder whose numerator is of order less than its denominator (i.e., $F(s) = R(s) + N(s)/D(s)$).

$$F(s) = \frac{s^3 + 2s^2 + 6s + 7}{s^2 + s + 5} \longrightarrow F(s) = s + 1 + \frac{2}{s^2 + s + 5}$$

Based on roots of $D(s)$ there are three cases:

Case 1: Roots of the Denominator of $F(s)$ Are Real and Distinct

Case 2: Roots of the Denominator of $F(s)$ Are Real and Repeated

Case 3: Roots of the Denominator of $F(s)$ Are Complex or Imaginary

Case 1: Real and Distinct Roots

$$F(s)=\frac{N(s)}{D(s)}=\frac{N(s)}{(s+p_1)(s+p_2)\cdots(s+p_i)\cdots(s+p_n)}$$

Note: Order of $N(s)$ is less than the order of $D(s)$.

$$=\frac{K_1}{(s+p_1)}+\frac{K_2}{(s+p_2)}+\cdots+\frac{K_i}{(s+p_i)}+\cdots+\frac{K_n}{(s+p_n)}$$

K_i is constant and called Residue.

$$K_i=(s+p_i)F(s)\Big|_{s\to -p_i}=\frac{\cancel{(s+p_i)}N(s)}{(s+p_1)(s+p_2)\cdots\cancel{(s+p_i)}\cdots(s+p_n)}\Bigg|_{s\to -p_i}, i=1,\dots,n$$

$$\Rightarrow \quad f(t)=\mathcal{L}^{-1}\{F(s)\}=K_1e^{-p_1t}+\cdots+K_ie^{-p_it}+\cdots+K_ne^{-p_nt} \quad \text{for} \quad t\geq 0$$

Example

$$F(s) = \frac{2}{(s+1)(s+2)}$$

$$\mathcal{L}^{-1}\{F(s)\} = ?$$

Answer:

$$f(t) = 2e^{-t} - 2e^{-2t} \qquad t \geq 0$$

Case 2: Real and Repeated Roots

$$F(s)=\frac{N(s)}{D(s)}=\frac{N(s)}{(s+p_1)^r(s+p_2)\cdots(s+p_n)}$$

Note: Order of $N(s)$ is less than the order of $D(s)$.

$$=\frac{K_1}{(s+p_1)^r}+\frac{K_2}{(s+p_1)^{r-1}}+\cdots+\frac{K_r}{(s+p_1)}+\frac{K_{r+1}}{(s+p_2)}+\cdots+\frac{K_n}{(s+p_n)}$$

(Each multiple root generates additional terms consisting of denominator factors of reduced multiplicity)

- $K_1, K_{r+1}, \ldots, K_n$ can be found using the method explained in **Case 1**.
- $K_2, \ldots, K_r$ can be found using: $$K_i=\frac{1}{(i-1)!}\left.\frac{d^{i-1}\{(s+p_1)^rF(s)\}}{ds^{i-1}}\right|_{s\to -p_1},\quad i=2,\ldots,r$$

Note: For finding $f(t)$, we know $\mathcal{L}^{-1}\left\{\frac{1}{(s+a)^n}\right\}=e^{-at}\frac{t^{n-1}}{(n-1)!}$

Example

$$F(s) = \frac{2}{(s+1)(s+2)^2}$$

$$\mathcal{L}^{-1}\{F(s)\} = ?$$

Answer:

$$f(t) = 2e^{-t} - 2te^{-2t} - 2e^{-2t} \qquad t \geq 0$$

Case 3: Complex or Imaginary Roots
Method 1

$$F(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s+p_1)(s^2+as+b)\cdots}$$

$$= \frac{K_1}{(s+p_1)} + \frac{(K_2 s + K_3)}{(s^2+as+b)} + \cdots \quad (*)$$

Note: Order of $N(s)$ is less than the order of $D(s)$.

K_1 can be found using the method explained in **Case 1**.

K_2 and K_3 can be found by multiplying both sides of equation (*) by $D(s)$ and balancing coefficients of both sides of equation:

$$\frac{N(s)}{D(s)} = \frac{K_1}{(s+p_1)} + \frac{(K_2 s + K_3)}{(s^2+as+b)} + \cdots \Rightarrow N(s) = K_1(s^2+as+b) + (K_2 s + K_3)(s+p_1) + \cdots$$

Note: For finding $f(t)$, we know

$$s^2+as+b = \left(s+\frac{a}{2}\right)^2 + b - \left(\frac{a}{2}\right)^2 = (s+\sigma)^2+\omega^2,$$

$$\mathcal{L}^{-1}\left\{\frac{A(s+\sigma)+B\omega}{(s+\sigma)^2+\omega^2}\right\} = Ae^{-\sigma t}\cos\omega t + Be^{-\sigma t}\sin\omega t$$

Example

$$F(s) = \frac{3}{s(s^2 + 2s + 5)}$$

$$\mathcal{L}^{-1}\{F(s)\} = ?$$

Answer:

$$f(t) = \frac{3}{5} - \frac{3}{5}e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right) \qquad t \geq 0$$

Case 3: Complex or Imaginary Roots Method 2 (optional)

The techniques described for real roots, i.e., **Case 1** and **Case 2**, can be also used for complex and imaginary roots.

$$F(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s+p_1)(s^2+as+b)\cdots} = \frac{K_1}{(s+p_1)} + \frac{K_2}{(s+\sigma+j\omega)} + \frac{K_3}{(s+\sigma-j\omega)} + \cdots$$

- K_1 and K_2 can be found using the method explained in **Case 1**, and K_3 will be the **complex conjugate** of K_2.
- Using this general method, inverse Laplace transform of a function with Repeated Complex or Imaginary Roots can be also found using the method explained in **Case 2**.

Note: For finding $f(t)$, we know

$$\begin{cases} \dfrac{e^{j\omega t}+e^{-j\omega t}}{2} = \cos\omega t \\ \dfrac{e^{j\omega t}-e^{-j\omega t}}{2j} = \sin\omega t \end{cases}$$

Example

$$F(s) = \frac{3}{s(s^2+2s+5)} \qquad \mathcal{L}^{-1}\{F(s)\} = ?$$

$$F(s) = \frac{3}{s(s^2+2s+5)} = \frac{K_1}{s} + \frac{K_2}{s+1+j2} + \frac{K_3}{s+1-j2}$$

K_1, K_2, and K_3 can be found using method explained in Case 1:

K_3 is **complex conjugate** of K_2:

$$K_1 = \frac{3}{5}, \qquad K_2 = \left.\frac{3}{s(s+1-j2)}\right|_{s\to -1-j2} = -\frac{3}{20}(2+j1), \qquad K_3 = -\frac{3}{20}(2-j1)$$

$$F(s) = \frac{3}{5}\frac{1}{s} - \frac{3}{20}\left(\frac{2+j1}{s+1+j2} + \frac{2-j1}{s+1-j2}\right)$$

$$f(t) = \frac{3}{5} - \frac{3}{20}\left[(2+j1)e^{-(1+j2)t} + (2-j1)e^{-(1-j2)t}\right]$$

$$f(t) = \frac{3}{5} - \frac{3}{20}e^{-t}\left[4\left(\frac{e^{j2t}+e^{-j2t}}{2}\right) + 2\left(\frac{e^{j2t}-e^{-j2t}}{2j}\right)\right] \quad \Rightarrow \quad f(t) = \frac{3}{5} - \frac{3}{5}e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right)$$

$$t \geq 0$$

Example

$$G(s) = \frac{s^3 + 5s^2 + 9s + 7}{(s+1)(s+2)} \qquad \mathcal{L}^{-1}\{G(s)\} = ?$$

Answer:

$$g(t) = \frac{d}{dt}\delta(t) + 2\delta(t) + 2e^{-t} - e^{-2t} \qquad t \geq 0$$

Example

$$\mathcal{L}^{-1}\left\{\frac{s^2+2s+3}{(s+1)^3}\right\} = ?$$

Answer:

$$(1+t^2)e^{-t} \qquad t \geq 0$$

Example: Solving an ODE Using Laplace Transform

$$\ddot{x} + 2\dot{x} + 5x = 3, \qquad \dot{x}(0) = 0, \qquad x(0) = 0$$

Answer:

$$x(t) = \frac{3}{5} - \frac{3}{5}e^{-t}\cos 2t - \frac{3}{10}e^{-t}\sin 2t$$

Using MATLAB and Control System Toolbox

Laplace and Inverse Laplace Transforms Using laplace, ilaplace

```
syms s t w A a

F = laplace(A*exp(-a*t) * sin(w*t));
f = ilaplace(1/(s + 3)^2);
```

$$f(t) = Ae^{-at}\sin\omega t \qquad \mathcal{L}\{f(t)\} = ?$$

$$F(s) = \frac{1}{(s+3)^2} \qquad \mathcal{L}^{-1}\{F(s)\} = ?$$

Partial Fraction Expansion/Decomposition Using residue

[r,p,k] = residue(b,a) finds the residues, poles, and direct term of a Partial Fraction Expansion of the ratio of two polynomials:

$$\frac{b(s)}{a(s)}=\frac{b_m s^m+b_{m-1}s^{m-1}+\cdots+b_1 s+b_0}{a_n s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0}=\frac{r_n}{s-p_n}+\cdots+\frac{r_2}{s-p_2}+\frac{r_1}{s-p_1}+k(s)$$

where b = [bm ... b1 b0] and a = [an ... a1 a0] are coefficients of the polynomials, r = [rn ... r2 r1] are the residues, p = [pn ... p2 p1] are the poles, and k is a polynomial.

[b,a] = residue(r,p,k) converts the partial fraction expansion back to the ratio of two polynomials and returns the coefficients in b and a.

Example: $F(s)=\frac{-4s+8}{s^2+6s+8}=\frac{-12}{s+4}+\frac{8}{s+2}$

```
b = [-4 8];
a = [1 6 8];
[r,p,k] = residue(b,a)
[b,a] = residue(r,p,k)
```

Output of `[r,p,k] = residue(b,a)`:

```
r = [-12; 8]
p = [-4; -2]
k = []
```

Output of `[b,a] = residue(r,p,k)`:

```
b = [-4 8]
a = [1 6 8]
```