Ch3: Modeling of Dynamic Systems – Part 1

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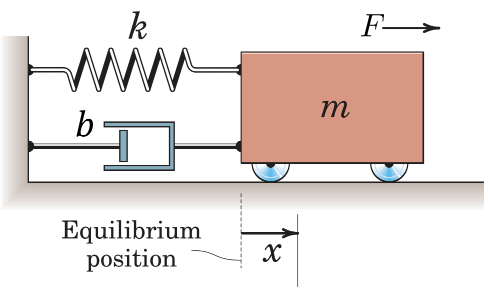
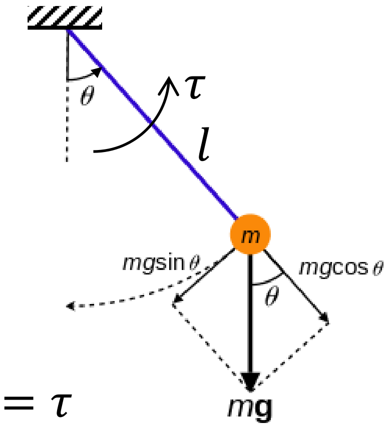
Introduction

Mathematical Modeling

As the first part of control systems analysis, the dynamic systems (i.e., mechanical, electrical, thermal, economic, biological, ...) must be **mathematically modeled** in terms of **differential equations** using **physical laws** (e.g., Newton's laws for mechanical systems and Kirchhoff's laws for electrical systems).



$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F$$



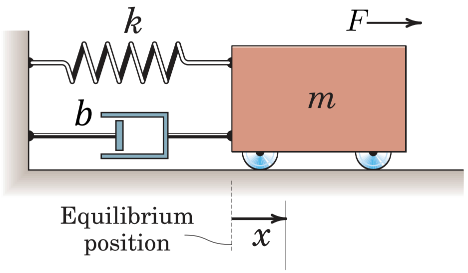
$$ml^2\frac{d^2\theta}{dt^2} + mgl\sin\theta = \tau$$

Mathematical Modeling: Simplicity vs. Accuracy

In obtaining a mathematical model, we must make a **compromise** between the **simplicity** of the model and the **accuracy** of the results of the analysis. We may simplify the system model in order to design a relatively simple controller.

Sometimes, it is necessary to ignore some nonlinearities and distributed parameters of a system to derive a **linear lumped-parameter model** and represent it using Ordinary Differential Equation (ODE) instead of a Partial Differential Equation (PDE) .

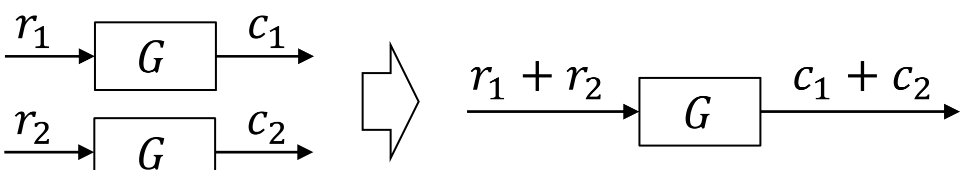
$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F$$



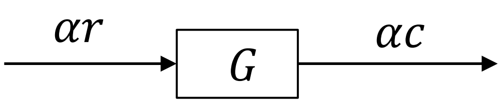
However, a linear lumped-parameter model, which may be valid in **low-frequency** operations, may not be valid at sufficiently high frequencies, (due to presence of neglected distributed parameters). For example, The mass of a spring may be neglected in low frequency operations, but it becomes an important property of the system at high frequencies.

Linear Systems

A system is called **Linear** with respect to its inputs and outputs if it satisfies two properties **Superposition** and **Homogeneity**.



(Superposition)



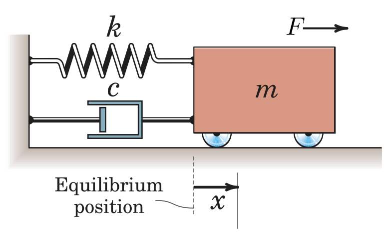
(Homogeneity)

$\alpha \in \mathbb{R}$

- A **differential equation** is linear if the **coefficients** are **constants** or **functions only of the independent variable**.

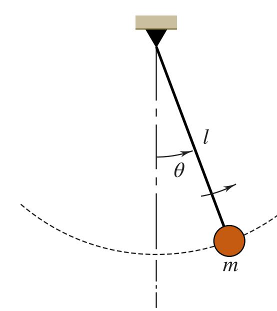
$$a_n(t)c^{(n)} + \cdots + a_2(t)\ddot{c} + a_1(t)\dot{c} + a_0(t)c = r(t)$$

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F$$



(Linear System)

$$ml^2\frac{d^2\theta}{dt^2} + mgl\sin\theta = T$$



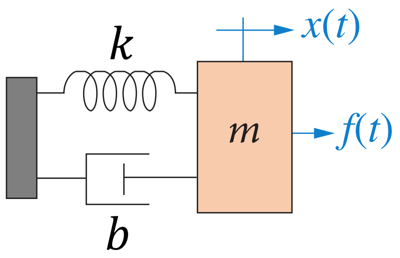
(Nonlinear System)

Time-Invariant vs. Time Varying Systems

A system is said to be **Time-Invariant** if the relationship between the input and output is **independent** of time.

- If the response to $r(t)$ is $c(t)$, then the response to $r(t-t_0)$ is $c(t-t_0)$.

Ex. A mass-spring-damper system which its physical parameters remains constant.



A system is said to be **Time-Varying/Varient** if the relationship between the input and output is **dependent** of time.

Ex. A spacecraft system which its mass changes due to fuel consumption.

LTI: Linear Time-Invariant
LTV: Linear Time-Varying

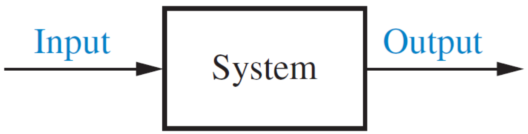
System Representation

The ODE is not a satisfying representation because the system input $r(t)$ and output $c(t)$ appear throughout the equation.

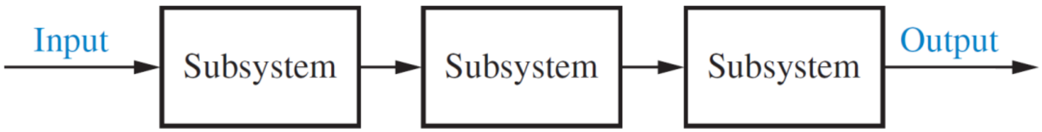
$$\frac{d^n c(t)}{dt^n} + a_{n-1}\frac{d^{n-1} c(t)}{dt^{n-1}} + \cdots + a_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1}\frac{d^{m-1} r(t)}{dt^{m-1}} + \cdots + b_0 r(t)$$

$$n \geq m$$

It is preferred a mathematical representation which the input, output, and system are **separate parts**, and it can be modeled as a block diagram.



or

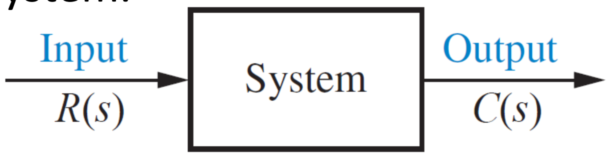


(cascaded interconnections of several subsystems)

- Two methods for representation of mathematical models of dynamic systems:
  (1) **Transfer Function (TF)** in the **Frequency Domain**,
  (2) **State-Space (SS) Representation** in the **Time Domain**.

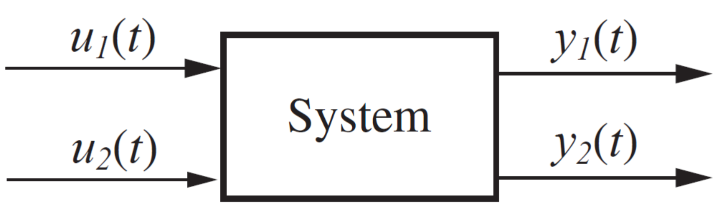
Classical vs. Modern Control Theory

Classical (or **Frequency-Domain** or **Transfer-Function**) approach [Since 1920s] can be applied only to **linear**, **time-invariant** (**LTI**), **SISO** (**Single-Input Single-Output**) systems with **zero initial conditions**, or systems that can be approximated as such. It does not use any knowledge of the **interior structure** of the system.



Transfer Function

Modern (or **Time-Domain** or **State-Space**) approach [Since 1960s] can be applied to a wide range of systems including **nonlinear**, **time variant** (**non-autonomous**), **MIMO** (**Multi-Input Multi-Output**) systems with **nonzero initial conditions** and also **LTI systems** modeled by the classical approach.



State Space

Transfer Function

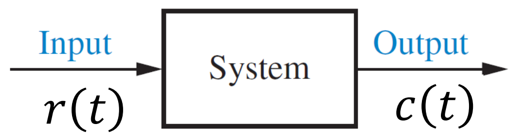
Transfer Function

Consider a general nth-order, linear time-invariant (LTI) differential equation as

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \cdots + a_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \cdots + b_0 r(t)$$

$$n \geq m$$

where $c(t)$ is the output and $r(t)$ is the input.



By taking the Laplace transform:

$$a_n s^n C(s) + a_{n-1} s^{n-1} C(s) + \cdots + a_0 C(s) + \text{initial condition terms involving } c(t) =$$

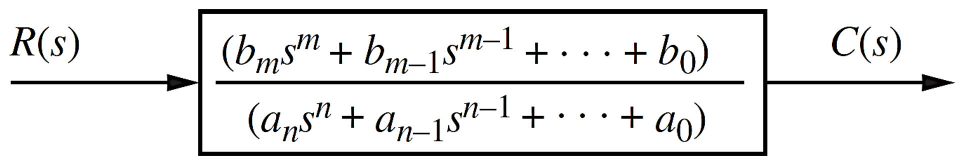
$$b_m s^m R(s) + b_{m-1} s^{m-1} R(s) + \cdots + b_0 R(s) + \text{initial condition terms involving } r(t)$$

Transfer Function

If we assume that **all initial conditions** are **zero**, the ratio of the output transform $C(s)$, divided by the input transform $R(s)$ is called **Transfer Function (TF)**:

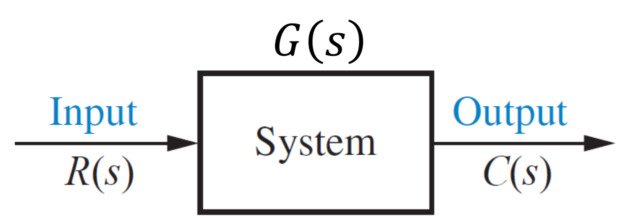
$$\frac{C(s)}{R(s)} = G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0} = \frac{\mathcal{L}[\text{Output}]}{\mathcal{L}[\text{Input}]}\Bigg|_{\text{zero initial conditions}}$$

It is a function that **algebraically** relates a system's output to its input.



Thus, the output $C(s)$, the input $R(s)$, and the system $G(s)$ are now separated.

$$C(s) = G(s)R(s)$$



Comments on Transfer Function

- The TF does not provide any information concerning the physical structure of the system. The TFs of many physically different systems can be identical.

- If the TF of a system is unknown, it may be established experimentally by introducing known inputs and studying the output of the system.

- The response of an LTI system to a unit-impulse input ($R(s) = 1$) when the initial conditions are zero is:

$$C(s) = G(s)R(s) = G(s) \quad \rightarrow \quad \mathcal{L}^{-1}[C(s)] = \mathcal{L}^{-1}[G(s)] = g(t)$$

  Therefore, the impulse-response function of an LTI system contain information about the system dynamics. Hence, it is possible to obtain complete information about the dynamic characteristics of the system by exciting it with an **impulse input** and measuring the response.

Example

Find the transfer function, then, find the response $c(t)$ to a unit step input of the following system, assuming zero initial conditions:

$$\frac{dc(t)}{dt} + 2c(t) = r(t)$$

Answer:

$$c(t) = \frac{1}{2} - \frac{1}{2}e^{-2t}$$

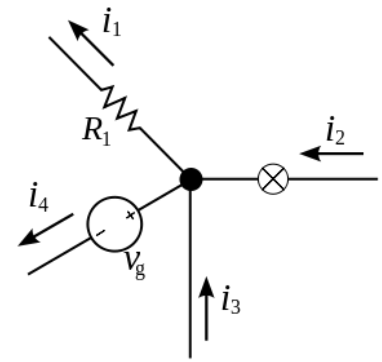
Electrical Systems

Laws Governing Electrical Circuits

Basic laws governing electrical circuits are **Kirchhoff's Current and Voltage Laws**.

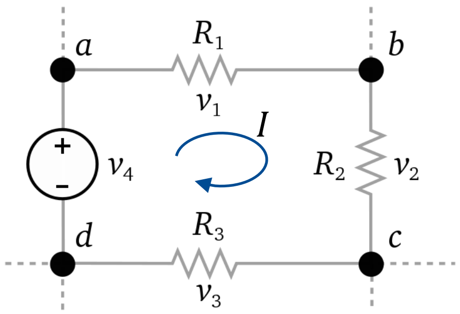
Kirchhoff's Current Law (KCL) or **Node Law**: The algebraic sum of all currents entering and leaving a node is zero (or the sum of currents entering a node is equal to the sum of currents leaving the same node).

$$i_2 + i_3 = i_1 + i_4$$



Kirchhoff's Voltage Law (KVL) or **Loop Law**: The algebraic sum of the voltages around any loop in an electrical circuit is zero (or the sum of the voltage drops is equal to the sum of the voltage rises around a loop).

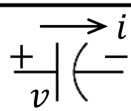
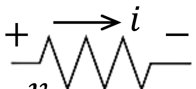
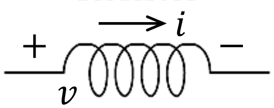
$$v_4 - v_1 - v_2 - v_3 = 0$$



A mathematical model of an electrical circuit can be obtained by applying one or both of Kirchhoff's laws to it.

Passive Linear Electrical Components

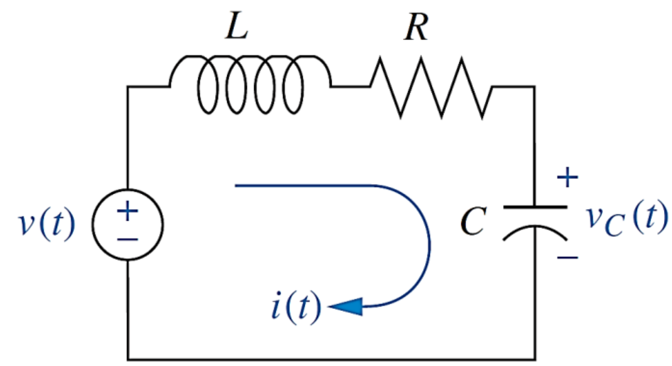
Passive linear components: **Resistors**, **Capacitors**, and **Inductors**.

Component	Voltage-current	Current-voltage
 Capacitor	$v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$	$i(t) = C\frac{dv(t)}{dt}$
 Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$
 Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^t v(\tau)d\tau$

$v(t)$ (volt, V), $i(t)$ (amp, A), C (farad, F), R (ohm, Ω), L (henry, H)
(voltage) (current) (capacitance) (resistance) (inductance)

Example: Transfer Function of an RLC Circuit

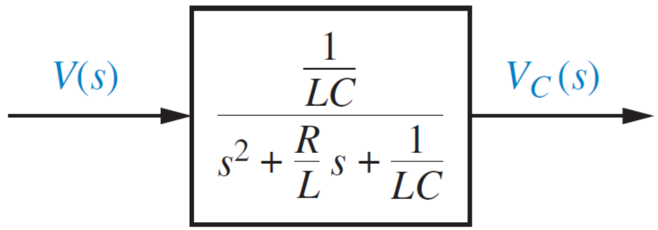
$$\frac{V_c(s)}{V(s)} = ?$$



Solution:

$$\begin{cases} L\dfrac{di(t)}{dt} + Ri(t) + \dfrac{1}{C}\displaystyle\int_0^t i(\tau)d\tau = v(t) & \longrightarrow \quad LsI(s) + RI(s) + \dfrac{1}{C}\dfrac{1}{s}I(s) = V(s) \\ \dfrac{1}{C}\displaystyle\int_0^t i(\tau)d\tau = v_C(t) & \longrightarrow \quad \dfrac{1}{C}\dfrac{1}{s}I(s) = V_c(s) \end{cases}$$

$$\frac{V_c(s)}{V(s)} = \frac{1}{LCs^2 + RCs + 1}$$

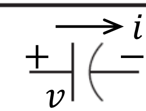
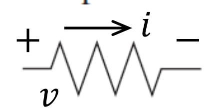
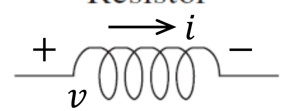


Impedance of Electrical Components

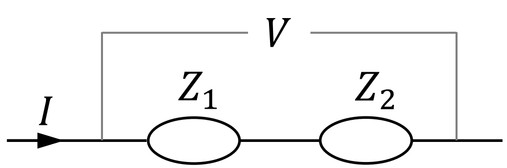
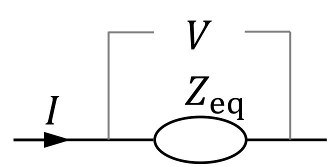
Impedance of electrical elements is defined as:

$$Z(s) = \frac{V(s)}{I(s)}$$

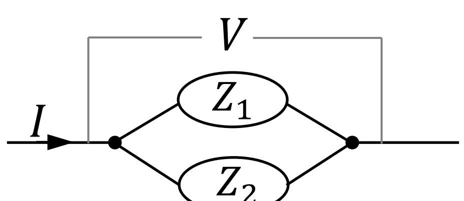
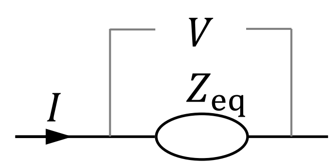
(It is like the definition of resistance.)

Component	Impedance $Z(s) = V(s)/I(s)$
 Capacitor	$\frac{1}{Cs}$
 Resistor	R
 Inductor	Ls

Equivalent impedance of elements in series and parallel:

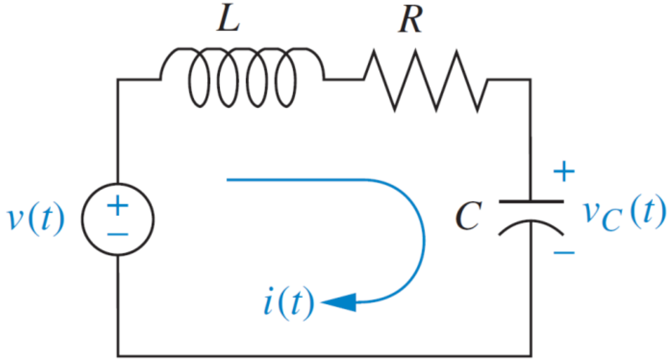
$$Z_{eq} = Z_1 + Z_2$$ (Series)

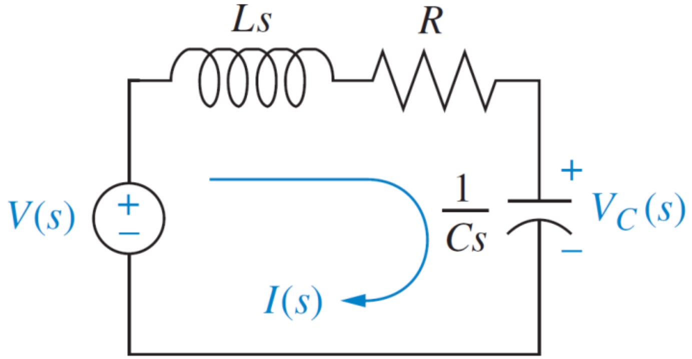
$$\frac{1}{Z_{eq}} = \frac{1}{Z_1} + \frac{1}{Z_2}$$ (Parallel)

Transfer Function of an RLC Circuit Using Impedance Approach

The concept of **impedance** simplifies the derivation of the transfer function of systems, without writing the differential equations.



Transformed Circuit



By applying Kirchhoff's voltage law to the transformed circuit:

$$\left(Ls + R + \frac{1}{Cs}\right)I(s) = V(s)$$

[Sum of impedances]$I(s)$= [Sum of applied voltages]

For capacitor:

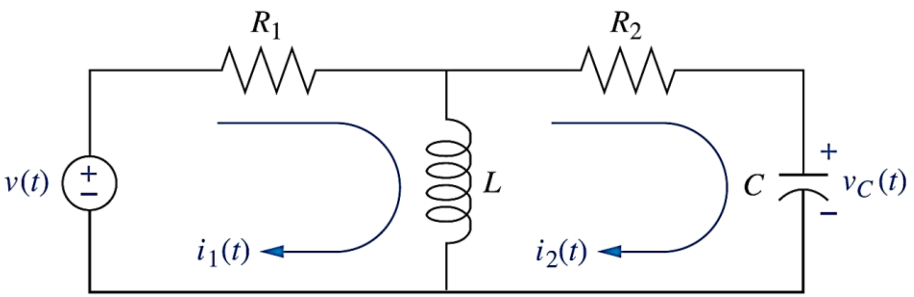
$$V_C(s) = I(s)\frac{1}{Cs}$$

$$\frac{V_C(s)}{V(s)} = \frac{1}{LCs^2 + RCs + 1}$$

Note: The impedance approach is valid only if **the initial conditions** involved are all **zeros**.

Example

Find $\frac{I_2(s)}{V(s)}$ using the impedance approach.

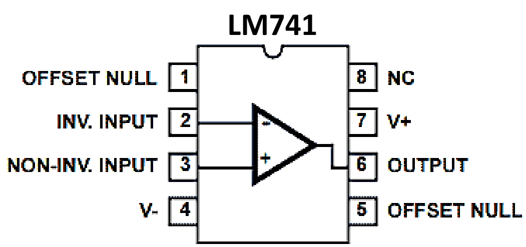


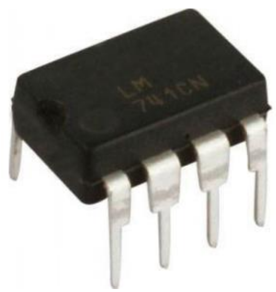
Answer:

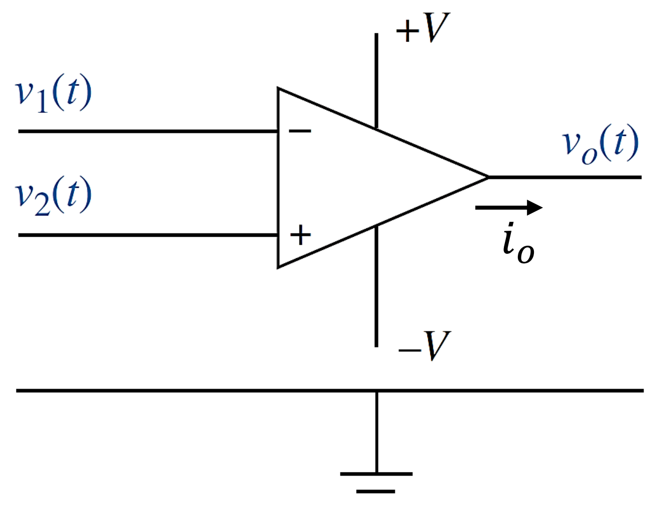
$$\frac{I_2(s)}{V(s)} = \frac{LCs^2}{(R_1 + R_2)LCs^2 + (R_1R_2C + L)s + R_1}$$

Operational Amplifiers

Operational Amplifiers (**Op Amps**) are active electrical components and are usually used in sensor circuits for amplifying signals or in filters for compensation purposes.







Characteristics of Op Amps:

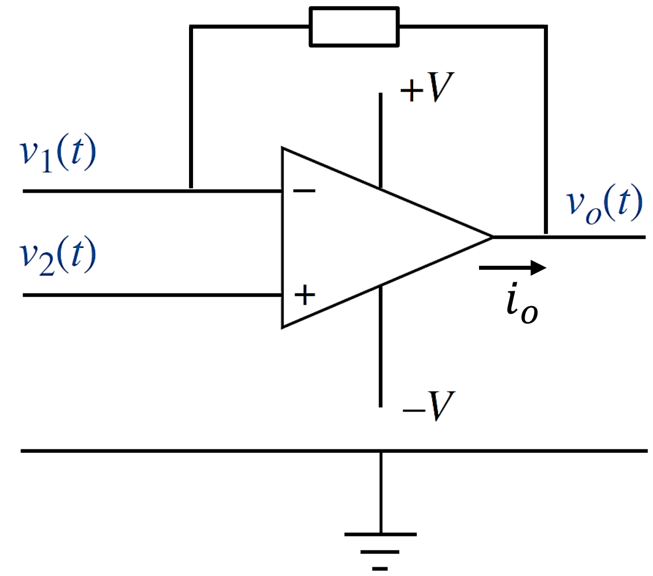
1. The input and output voltages are measured **relative to the ground** which its voltage is 0.

2. The input v_1 to the minus terminal of the amplifier is **inverted**, and the input v_2 to the plus terminal is **not inverted**. The total input to the amplifier thus becomes $v_2 - v_1$.

Operational Amplifiers

3. Output is $v_o = K(v_2 - v_1)$ where K is the differential or voltage **gain** and its magnitude is approximately ∞ (the op amp amplifies the difference in voltages v_1 and v_2)

4. Since the gain K of the op amp is very high, it is necessary to have a feedback from the **output to the inverted input** (negative feedback) to make the amplifier **stable**.



5. In the **ideal** op amp, (1) no current flows into the input terminals, and (2) the output voltage v_o is not affected by the load connected to the output terminal. In other words, the input impedance is infinity ($Z_i = \infty$) and the output impedance is zero ($Z_o = 0$).

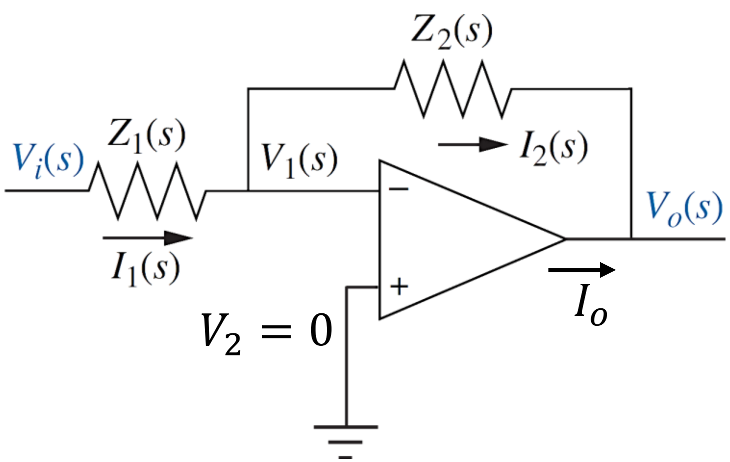
Two common configurations of Op Amps are **Inverting** and **Noninverting**.

Inverting Amplifier

If $v_2(t)$ is grounded, the amplifier is called an **inverting operational amplifier**.

Since no current flows into the input terminals of Op Amps, then $I_1(s) = I_2(s)$:

$$\frac{V_i(s) - V_1(s)}{Z_1(s)} = \frac{V_1(s) - V_o(s)}{Z_2(s)}$$



(Transformed Circuit)

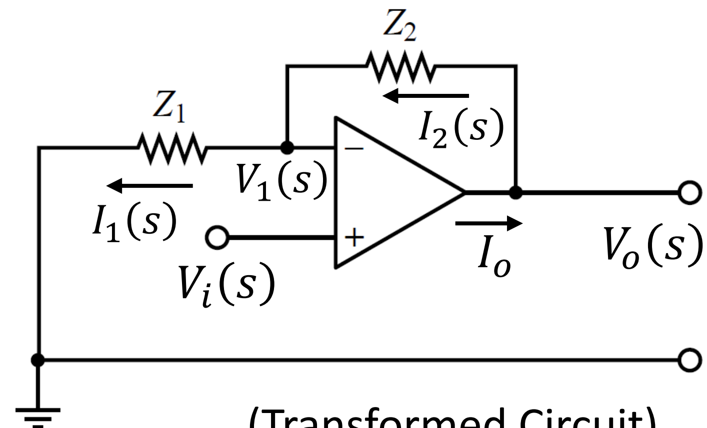
$$K(0 - V_1(s)) = V_o(s) \longrightarrow \frac{V_o(s)}{V_i(s)} = \frac{Z_2}{-Z_1 - (Z_1 + Z_2)/K}$$

Since $K \to \infty$, the transfer function of the inverting operational amplifier is

$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)}$$

Noninverting Amplifier

General **noninverting operational amplifier** circuit:



(Transformed Circuit)

Since no current flows into the input terminals of Op Amps, then $I_1(s) = I_2(s)$:

$$\frac{V_1(s) - 0}{Z_1(s)} = \frac{V_o(s) - V_1(s)}{Z_2(s)}$$

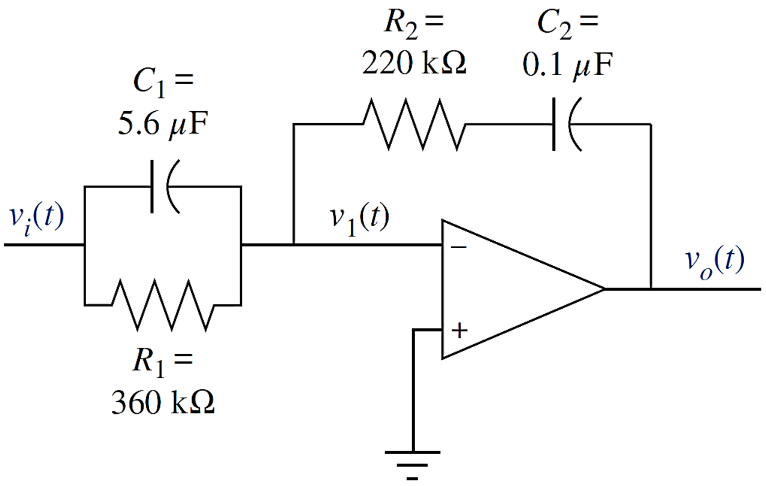
$$K\big(V_i(s) - V_1(s)\big) = V_o(s) \longrightarrow \frac{V_o(s)}{V_i(s)} = \frac{Z_1(s) + Z_2(s)}{Z_1(s) + \big(Z_1(s) + Z_2(s)\big)/K}$$

Since $K \to \infty$, the transfer function of the noninverting operational amplifier is

$$\frac{V_o(s)}{V_i(s)} = 1 + \frac{Z_2(s)}{Z_1(s)}$$

Example: PID Controller

$$\frac{V_o(s)}{V_i(s)} = ?$$

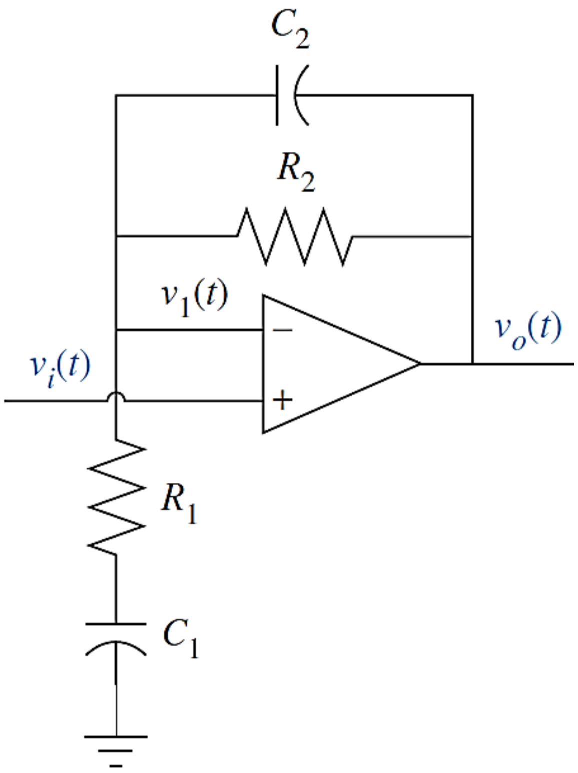


Answer:

$$\frac{V_o(s)}{V_i(s)} = -1.232\,\frac{s^2 + 45.95s + 22.55}{s}$$

Example

$$\frac{V_o(s)}{V_i(s)} = ?$$



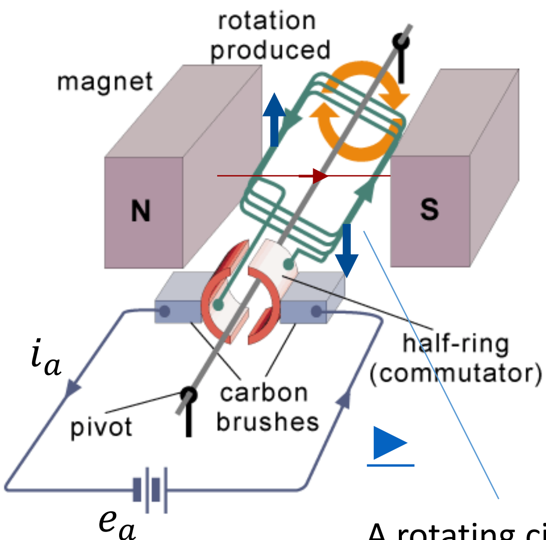
Answer:

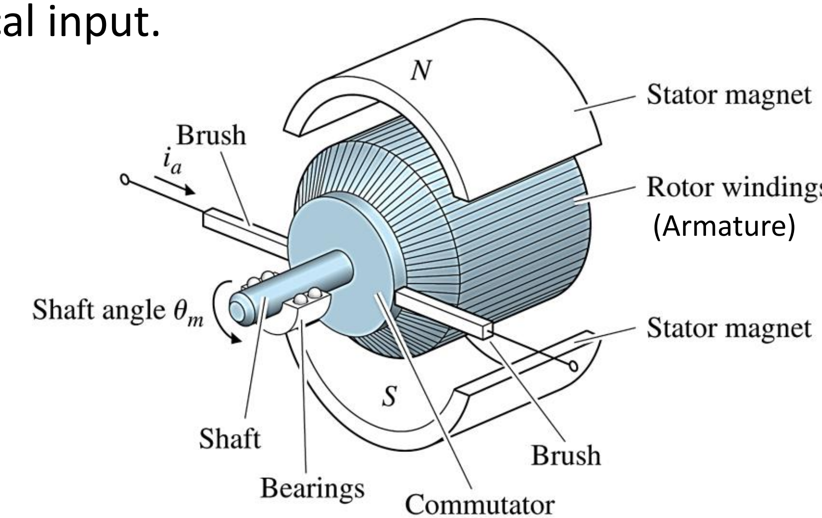
$$\frac{V_o(s)}{V_i(s)} = \frac{C_2C_1R_2R_1s^2 + (C_2R_2 + C_1R_2 + C_1R_1)s + 1}{C_2C_1R_2R_1s^2 + (C_2R_2 + C_1R_1)s + 1}$$

Electromechanical Systems

DC Motor

A motor is an **Electromechanical** system that provides a rotary motion for a voltage input, i.e., a mechanical output generated by an electrical input.





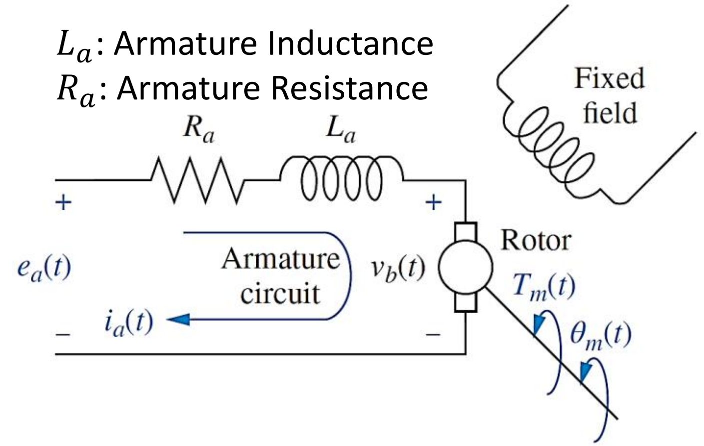
A rotating circuit called the armature, through which the current flows, passes through a constant magnetic field (developed by stationary permanent magnets) at right angles and produces a force which its resulting torque turns the rotor.

There are two ways to control DC Motors:

1. Adjusting the voltage across the armature terminals (**Armature Control**)
2. Adjusting the field flux (**Field Control**)

Armature-Controlled DC Motor

Electrical Equations:



1- Kirchhoff's Voltage Law (KVL):

Armature circuit equation:

$$e_a(t) = R_a i_a(t) + L_a \frac{di_a(t)}{dt} + v_b(t) \quad (1)$$

where e_a is the applied armature voltage, i_a is the armature current, and v_b is the back electromotive force (back EMF).

2- Voltage-Speed Relationship: Since the armature is rotating in a magnetic field, its back EMF voltage is proportional to angular velocity.

$$v_b(t) = K_b \dot{\theta}_m(t) = K_b \omega_m(t) \quad (2)$$

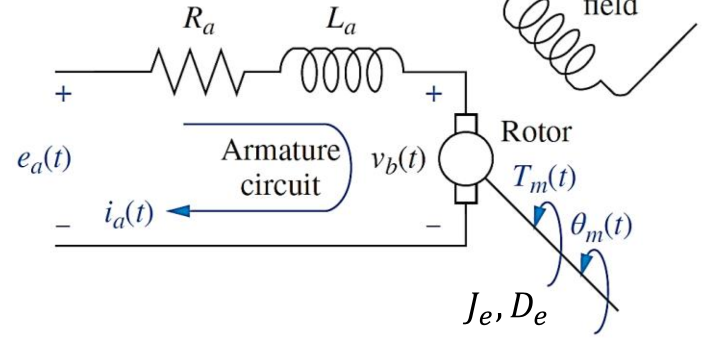
where K_b is the back EMF constant (unit: V·s/rad), and $\dot{\theta}_m(t) = \omega_m(t)$ is the angular velocity of the motor. $K_v = 1/K_b$ is called velocity (or speed) constant.

Armature-Controlled DC Motor

Mechanical Equation:

3- Newton's Law of Motion:

L_a: Armature Inductance
R_a: Armature Resistance



The relation between the torque T_m developed by the motor at the armature, equivalent inertia (J_e) at the armature, and equivalent viscous damping (D_e) at the armature is:

$$T_m(t) = J_e\ddot{\theta}_m(t) + D_e\dot{\theta}_m(t) \qquad (3)$$

4- Torque-Current Relationship: The torque developed by the motor is proportional to the armature current.

$$T_m(t) = K_t i_a(t) \qquad (4)$$

where K_t is the motor torque constant, which depends on the motor and magnetic field characteristics (unit: N·m/A). For ideal motor, $K_t = K_b$.

Armature-Controlled DC Motor

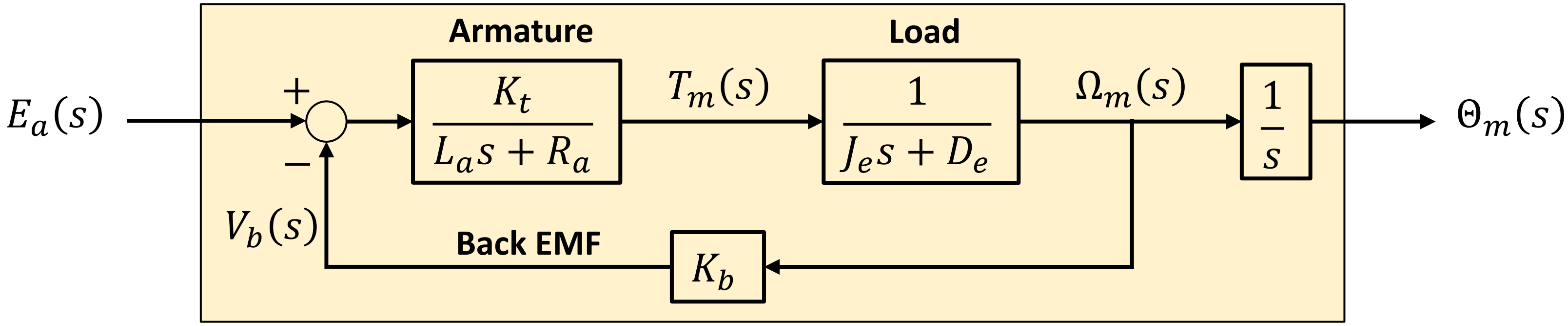
Thus, four equations represent the mathematical model of a DC motor:

(1) $e_a(t) = R_a i_a(t) + L_a \frac{di_a(t)}{dt} + v_b(t)$ (2) $v_b(t) = K_b \dot{\theta}_m$

(3) $T_m(t) = J_e \ddot{\theta}_m(t) + D_e \dot{\theta}_m(t)$ (4) $T_m(t) = K_t i_a(t)$

By considering e_a as input and θ_m as output, the transfer function of the motor with zero initial conditions is derived as:

$$\frac{\Theta_m(s)}{E_a(s)} = \frac{K_t}{s[(J_e s + D_e)(L_a s + R_a) + K_t K_b]}$$



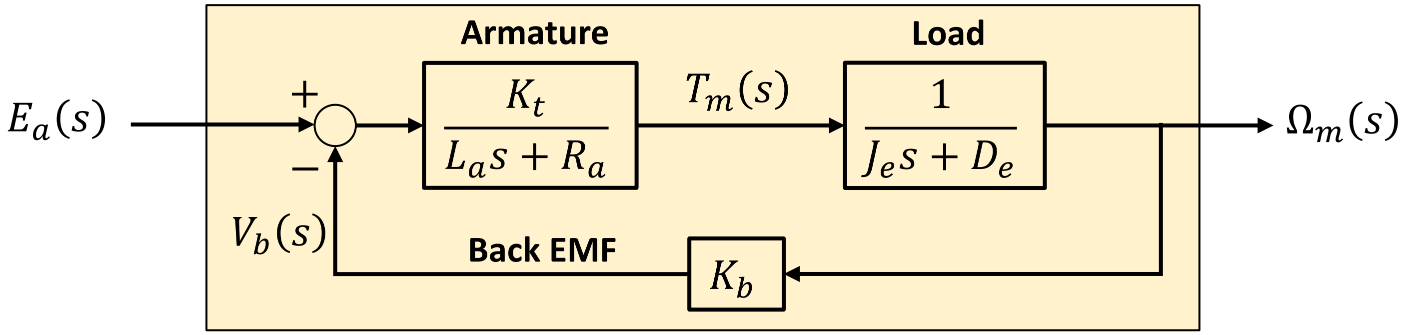
Armature-Controlled DC Motor

- It is usual for DC motors to assume that the armature inductance L_a is small compared to the armature resistance R_a (i.e., $L_a \approx 0$). Hence,

$$\frac{\Theta_m(s)}{E_a(s)} \approx \frac{K_t}{s[(J_e s + D_e)(R_a) + K_t K_b]} = \frac{K_t/(R_a J_e)}{s\left[s + \frac{1}{J_e}(D_e + \frac{K_t K_b}{R_a})\right]} = \frac{K}{s(s+\alpha)}$$

- In many cases, a transfer function between the applied armature voltage and the angular velocity of the motor is required. Hence,

$$\frac{\Omega_m(s)}{E_a(s)} = \frac{s\Theta_m(s)}{E_a(s)} \approx \frac{K_t}{[(J_e s + D_e)(R_a) + K_t K_b]} = \frac{K_t/(R_a J_e)}{\left[s + \frac{1}{J_e}(D_e + \frac{K_t K_b}{R_a})\right]} = \frac{K}{s+\alpha}$$



Measurement of Motor Constants

Mechanical Constants (J_e, D_e):

These constants can be determined through laboratory testing using **transient response** or **frequency response** data.

Electrical Constants ($K_t/R_a, K_b$):

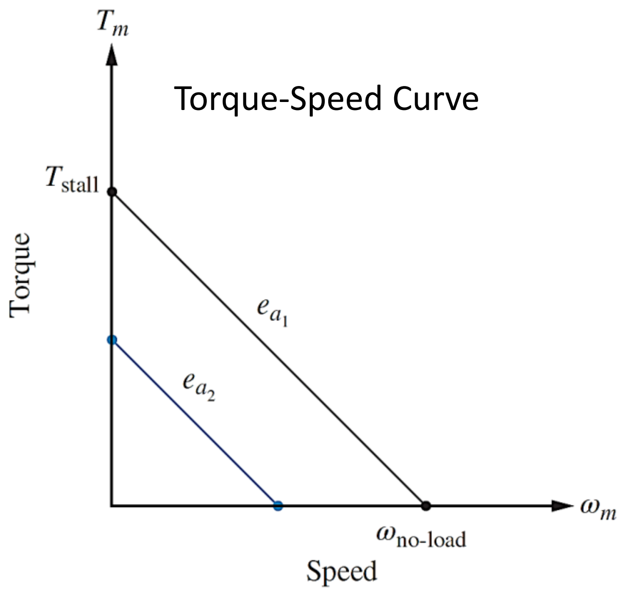
These constants can be obtained through a **dynamometer** test of the motor, where a **dynamometer** measures the **torque** and **speed** of a motor under the condition of a **constant applied voltage**.

Using (1), (2), (4), and assumption $L_a \approx 0$:

$$e_a = R_a \frac{T_m}{K_t} + K_b \omega_m \quad \xrightarrow{\text{or}} \quad T_m = -\frac{K_b K_t}{R_a}\omega_m + \frac{K_t}{R_a} e_a$$

T_m when $\omega_m = 0$ is called the Stall Torque (T_{stall}) and ω_m when $T_m = 0$ is called the no-load speed ($\omega_{\text{no-load}}$).

$$\Rightarrow \quad \frac{K_t}{R_a} = \frac{T_{\text{stall}}}{e_a} \qquad K_b = \frac{e_a}{\omega_{\text{no-load}}}$$



Mechanical Systems

Laws Governing Mechanical Systems

The fundamental law for obtaining the dynamic equations (equations of motion) of any mechanical system is Newton's second law:

$$\boldsymbol{F} = m\boldsymbol{a}_G$$

where $\boldsymbol{F}$ is the vector sum of all forces applied to each body in a system, $\boldsymbol{a}_G$ is the vector acceleration of each body with respect to an inertial reference frame, and m is the mass of the body.

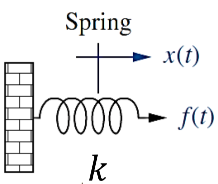
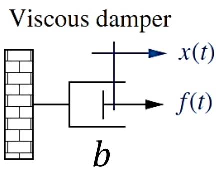
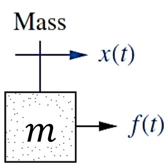
Application of Newton's law to one-dimensional **rotational** systems requires that the above equation be modified to:

$$M_G = I_G\alpha$$

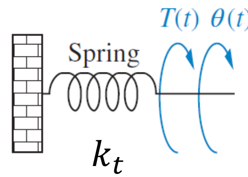
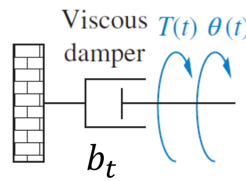
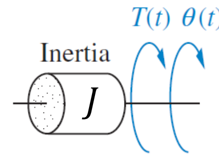
where M_G is the algebraic sum of all external moments about the center of mass of a body, I_G is the body's mass moment of inertia about its center of mass, and α is the angular acceleration of the body.

* Always use free-body diagram in applying Newton's laws!

Translational and Rotational Mechanical Systems

Component	Force-velocity	Force-displacement
	$f(t) = k\int_0^t v(\tau)d\tau$	$f(t) = kx(t)$
	$f(t) = bv(t)$	$f(t) = b\dfrac{dx(t)}{dt}$
	$f(t) = m\dfrac{dv(t)}{dt}$	$f(t) = m\dfrac{d^2x(t)}{dt^2}$

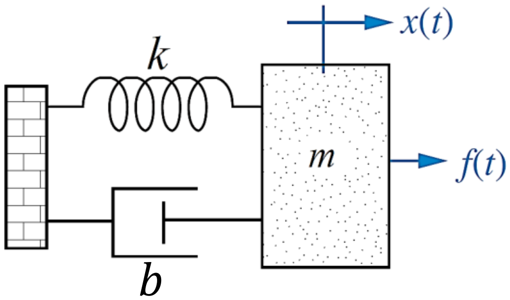
$f(t)$ (N), $x(t)$ (m), $v(t)$ (m/s),
k (N/m), b (N·s/m), m (kg)

Component	Torque-angular velocity	Torque-angular displacement
	$T(t) = k\int_0^t \omega(\tau)d\tau$	$T(t) = k\theta(t)$
	$T(t) = b_t\omega(t)$	$T(t) = b_t\dfrac{d\theta(t)}{dt}$
	$T(t) = J\dfrac{d\omega(t)}{dt}$	$T(t) = J\dfrac{d^2\theta(t)}{dt^2}$

$T(t)$ (N·m), $\theta(t)$ (rad), $\omega(t)$ (rad/s),
k_t (N·m/rad), b_t (N·m·s/rad), J (kg·m^2)

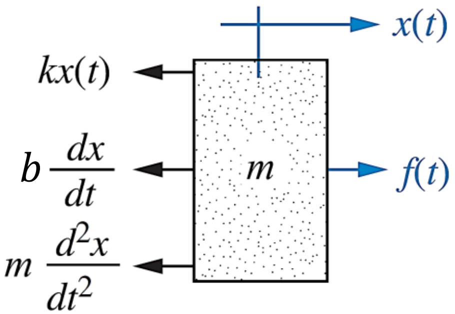
Example: Transfer Function of a Mass-Spring-Damper System

$$\frac{X(s)}{F(s)} = ?$$

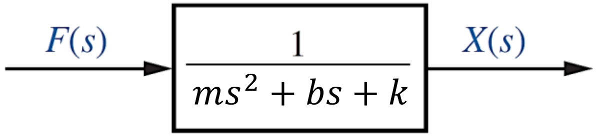


Solution:

$$m\ddot{x} + b\dot{x} + kx = f(t)$$



$$ms^2X(s) + bsX(s) + kX(s) = F(s) \longrightarrow \frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k}$$



Impedance of Mechanical Components

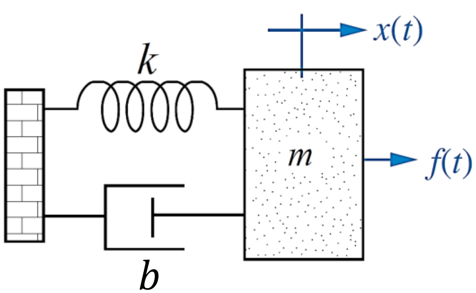
Impedance mechanical components is defined as:

$$Z(s) = \frac{F(s)}{X(s)}$$

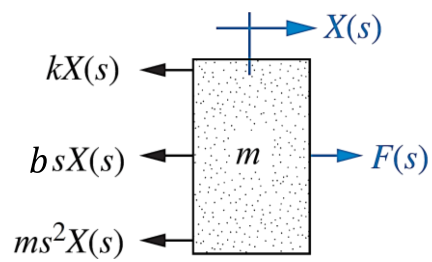
(It is like the definition of stiffness)

Component	Impedence $Z_M(s) = F(s)/X(s)$
Spring (k; $x(t)$, $f(t)$)	k
Viscous damper (b; $x(t)$, $f(t)$)	bs
Mass (m; $x(t)$, $f(t)$)	ms^2

Component	Impedence $Z_M(s) = T(s)/\theta(s)$
Spring (k_t; $T(t)$, $\theta(t)$)	k_t
Viscous damper (b_t; $T(t)$, $\theta(t)$)	$b_t s$
Inertia (J; $T(t)$, $\theta(t)$)	Js^2



Transformed System



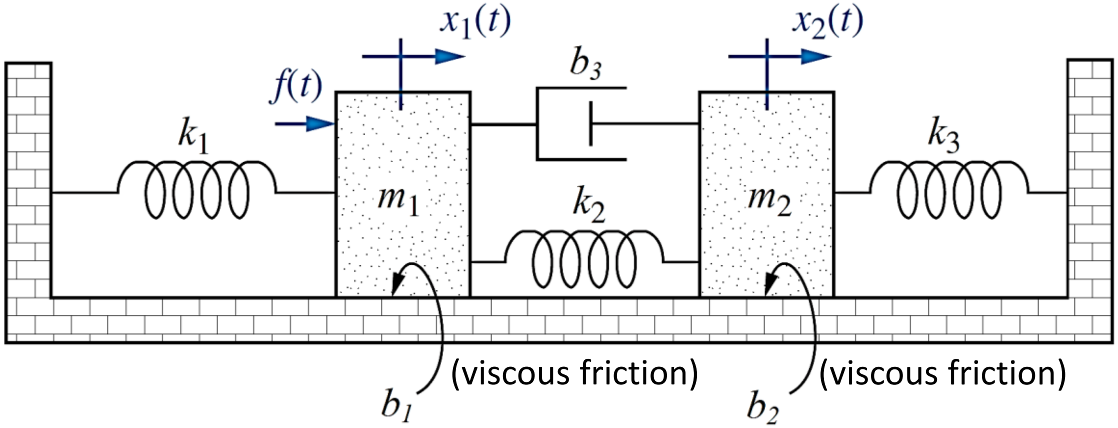
$$(ms^2 + bs + k)X(s) = F(s)$$

[Sum of impedances]$X(s)$
=[Sum of applied forces]

Note: The impedance approach is valid only if **the initial conditions** involved are all **zeros**.

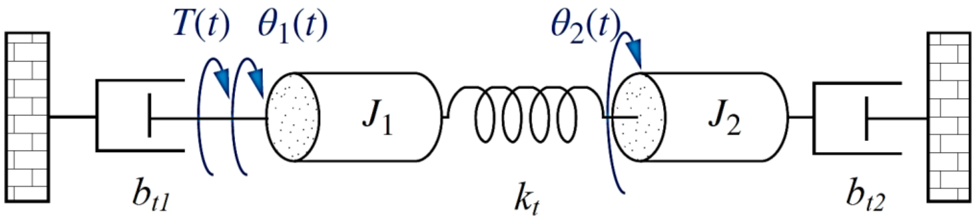
Example

Find $\dfrac{X_2(s)}{F(s)}$ using the transform method.



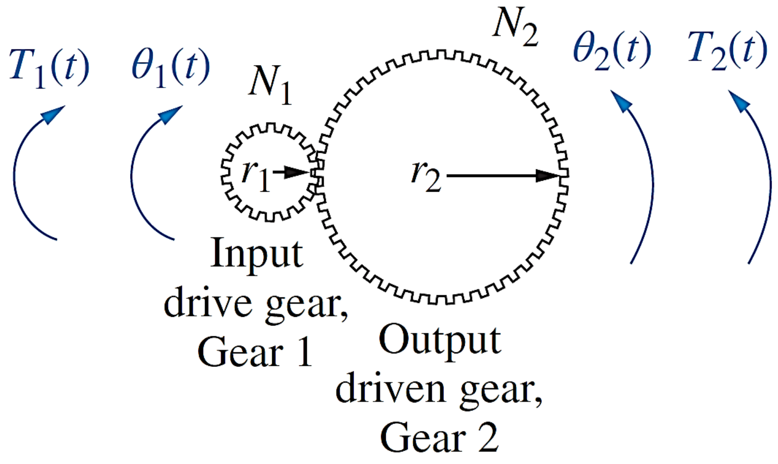
Example

$$\frac{\theta_2(s)}{T(s)} = ?$$



Transfer Function of Systems with Gears

The motors are often used in conjunction with gears to reduce the angular velocity and multiply the torque.



Consider a gear set including an input gear with radius r_1 and N_1 teeth and an output gear with radius r_2 and N_2 teeth.

$s_1 = s_2$ (traveled distance, no backlash) → $r_1\theta_1 = r_2\theta_2$

$m_1 = m_2$ (modulus: $2r/N$) → $\frac{2r_1}{N_1} = \frac{2r_2}{N_2}$

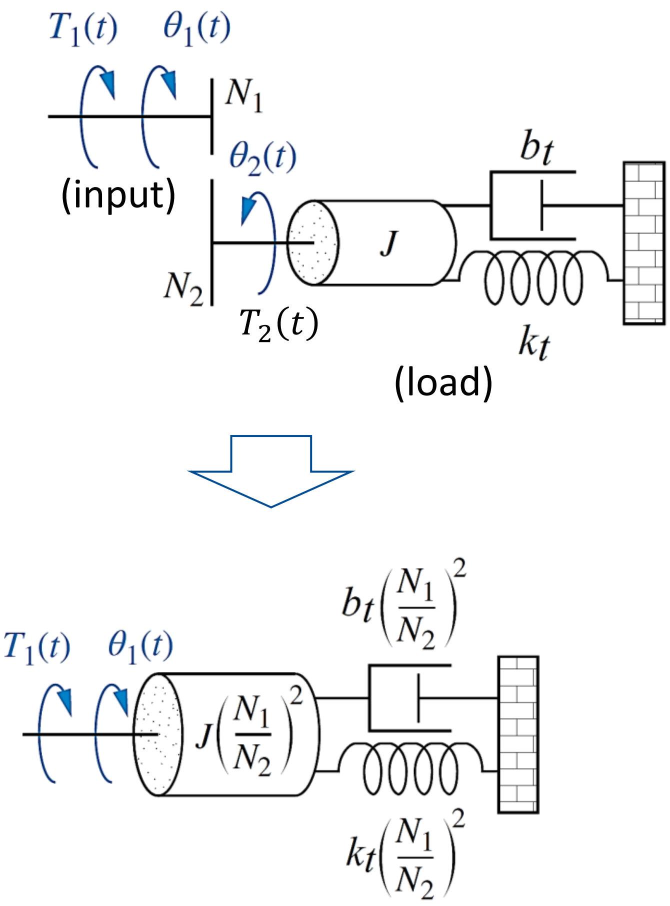
$E_1 = E_2$ (lossless gears) → $T_1\theta_1 = T_2\theta_2$

$$\frac{\theta_1}{\theta_2} = \frac{T_2}{T_1} = \frac{N_2}{N_1} = \frac{r_2}{r_1}$$

Example: Transfer Function of Systems with Gears

Consider the following system containing gears driving a rotational inertia, spring, and viscous damper. Find $\theta_1(s)/T_1(s)$.

$$\frac{\theta_1(s)}{T_1(s)} = ?$$



Solution:

$$(Js^2 + b_t s + k_t)\theta_2(s) = T_2(s) \quad \Rightarrow$$

$$(Js^2 + b_t s + k_t)\frac{N_1}{N_2}\theta_1(s) = T_1(s)\frac{N_2}{N_1} \quad \Rightarrow$$

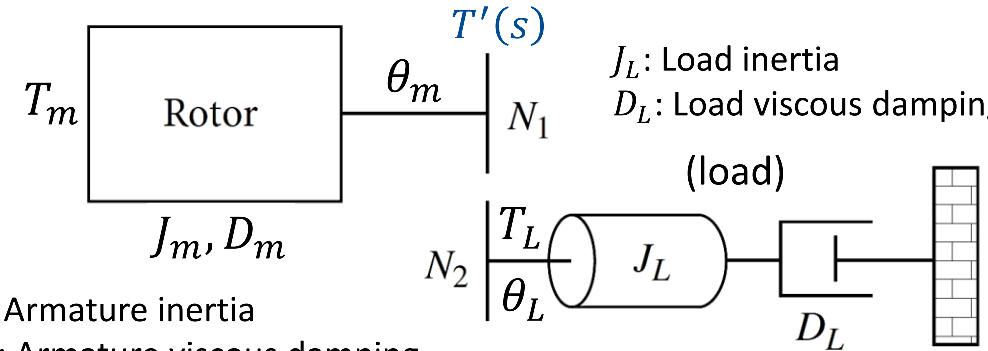
$$\left(J\left(\frac{N_1}{N_2}\right)^2 s^2 + b_t\left(\frac{N_1}{N_2}\right)^2 s + k_t\left(\frac{N_1}{N_2}\right)^2\right)\theta_1(s) = T_1(s)$$

In the equivalent system, gears are eliminated, and load is reflected from the output to the input.

Example

For the following system, find the equivalent inertia (J_e) and equivalent viscous damping (D_e) at the armature. Then, find the transfer function $\theta_m(s)/T_m(s)$. T_m is the torque developed by the motor.

$$\frac{\theta_m(s)}{T_m(s)} = ?$$



J_L: Load inertia
D_L: Load viscous damping

J_m: Armature inertia
D_m: Armature viscous damping

Solution:

$$T_m(s) - T'(s) = (J_m s^2 + D_m s)\theta_m(s)$$

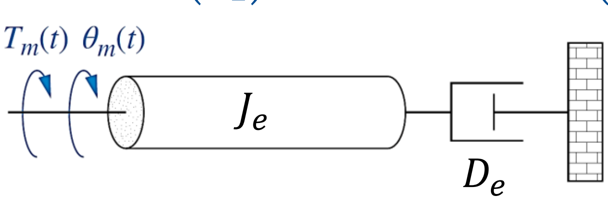
$$T_L(s) = (J_L s^2 + D_L s)\theta_L(s)$$

$$T'(s) = \left(\frac{N_1}{N_2}\right) T_L(s)$$

$$\theta_m(s) = \left(\frac{N_2}{N_1}\right) \theta_L(s)$$

$$T_m(s) = (J_e s^2 + D_e s)\theta_m(s)$$

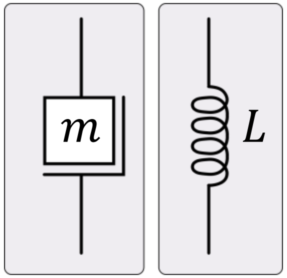
$$J_e = J_m + J_L \left(\frac{N_1}{N_2}\right)^2 ; D_e = D_m + D_L \left(\frac{N_1}{N_2}\right)^2$$



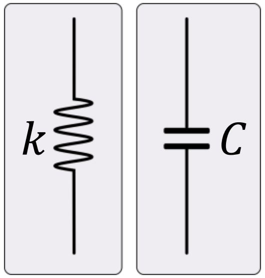
Mechanical-Electrical Analogies

There are **analogies** between **electrical** and **mechanical** components and variables.

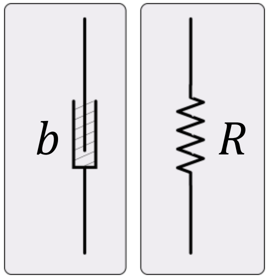
- The **mass** which is a kinetic energy-storage element is analogous to the **inductor** which is a magnetic field energy-storage element.



- The **spring** which is an elastic energy-storage element is analogous to the **capacitor** which is an electric field energy-storage element.



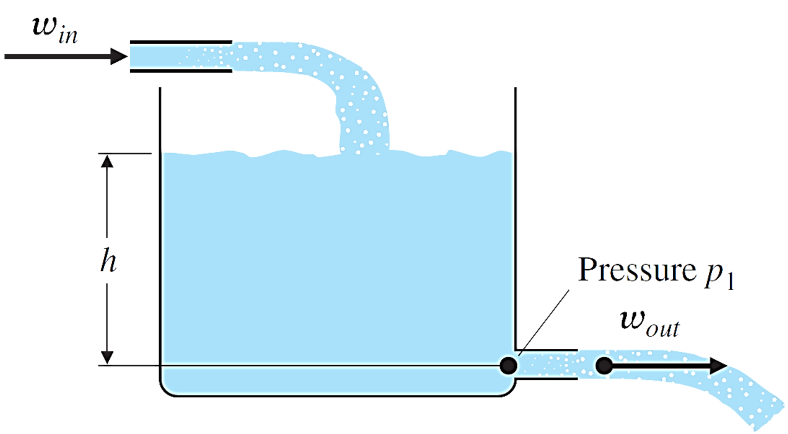
- The **viscous damper** which is a mechanical energy-dissipater element is analogous to the **resistor** which is an electrical energy-dissipater element.



Fluid Systems

Example

Determine the differential equation describing the height of the water in the tank.



w_{in}: Input mass flow rate

w_{out}: Output mass flow rate

A: Area of tank

ρ: Density of water

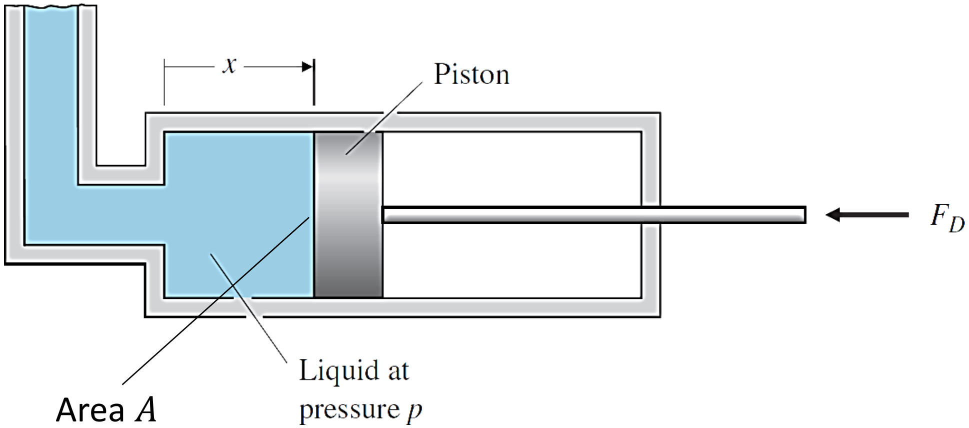
Answer:

$$\dot{h} = \frac{1}{A\rho}(w_{\text{in}} - w_{\text{out}})$$

Note: The physical relations governing fluid flow are continuity, force equilibrium, and resistance.

Example

Determine the differential equation describing the motion of the piston actuator, given that there is a force F_D acting on it and a pressure p in the chamber. Position of the piston is x and mass of the piston is M.



Answer:

$$M\ddot{x} = Ap - F_D$$