Ch7: Stability

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Stability Definition

Introduction

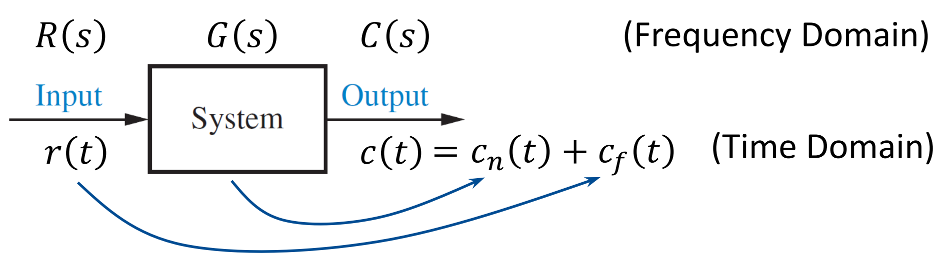
Stability is the most important system specification. **Instability** may have two causes:

1. The system being controlled may be unstable itself (Ex.: Segway).
2. Addition of feedback to the system may itself drive the system unstable.

The total time response $c(t)$ of a linear system is the sum of two responses:

1) Natural Response (or **homogeneous** solution) $c_n(t)$ which depends only on the system, not the input.

2) Forced Response (or **particular** solution) $c_f(t)$ which depends only on the input, not the system.

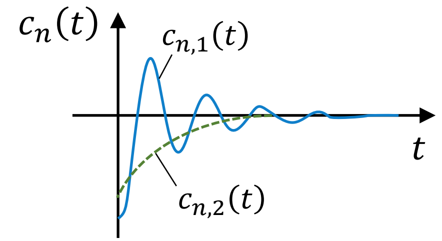


- Stability can be defined based on either the natural response $c_n(t)$ or the total response $c(t)$.

Definition of Stability Based on Natural Response c_n

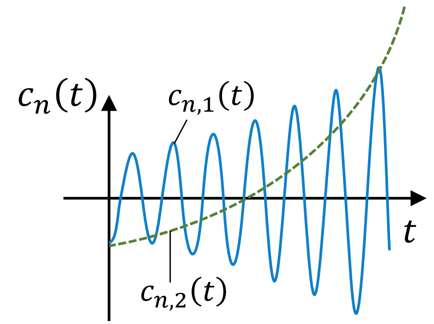
- An LTI system is **Stable** if the **natural response** approaches zero as time approaches infinity.

$$t \to \infty \quad \Rightarrow \quad c_n(t) \to 0$$

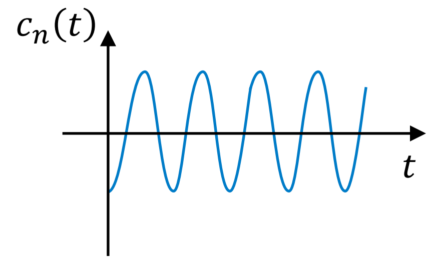


- An LTI system is **Unstable** if the **natural response** approaches infinity as time approaches infinity.

$$t \to \infty \quad \Rightarrow \quad c_n(t) \to \infty$$



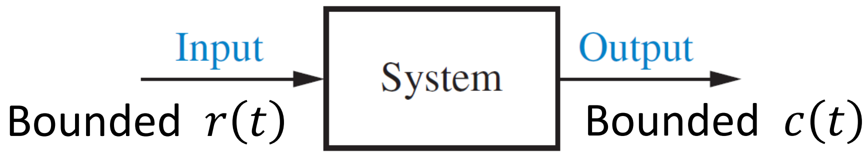
- An LTI system is **Marginally Stable** if the **natural response** remains constant or oscillates as time approaches infinity.

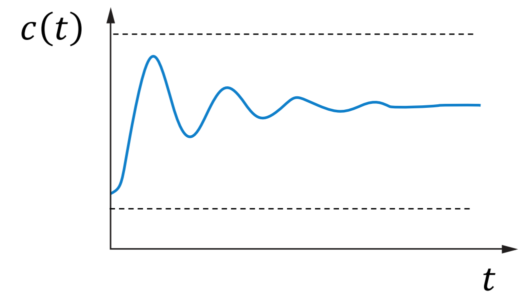


❖ These definitions rely on the **natural response**. However, sometimes, it is difficult to separate the natural response from the forced response by looking at the **total response**.

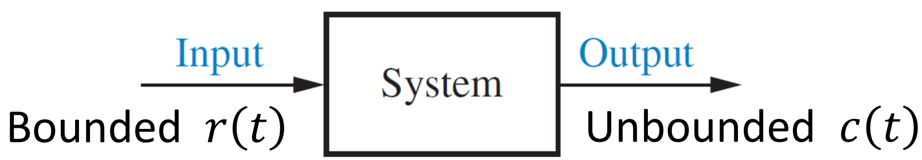
Definition of Stability Based on Total Response c (BIBO Stability)

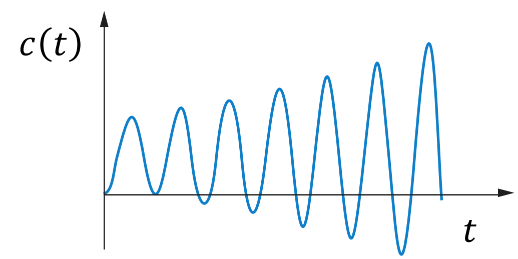
- A system is **Stable** if every bounded input yields a bounded output.





- A system is **Unstable** if any bounded input yields an unbounded output.





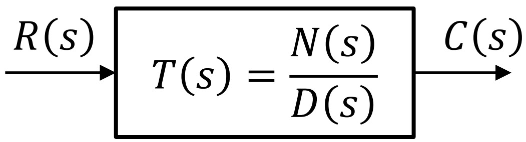
This definition is also called the **B**ounded-**I**nput **B**ounded-**O**utput (**BIBO**) **Stability**.

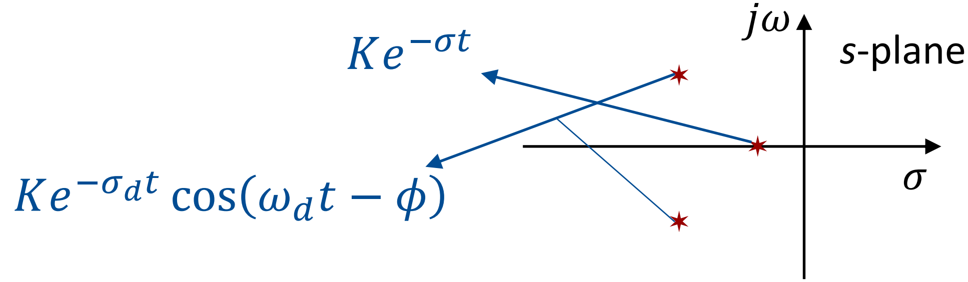
Note: If the **input is unbounded**, the **total response will be unbounded,** and it cannot be concluded whether the system is **stable or unstable**. Because it is not clear that the forced response is unbounded, or the natural response is unbounded.

Stability Determination

Stability Determination Using Poles: Stable System

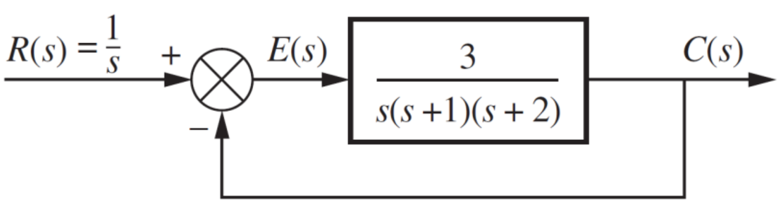
(1) If the closed-loop transfer function has **poles only in the left half-plane** (LHP), the system is **stable**.

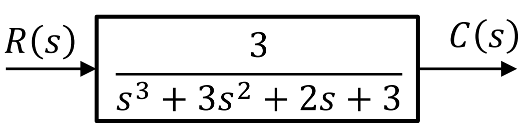


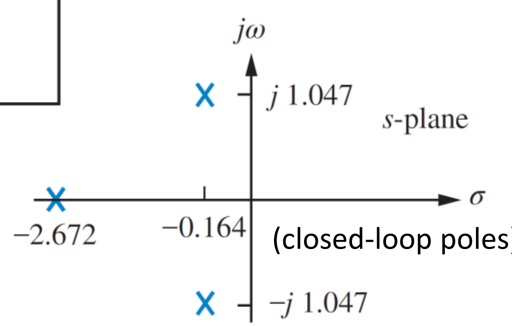


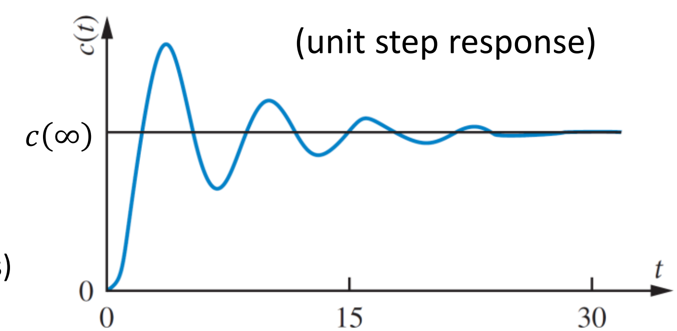
The natural response $c_n(t)$ decay to **zero** as time approaches **infinity**.

Example:



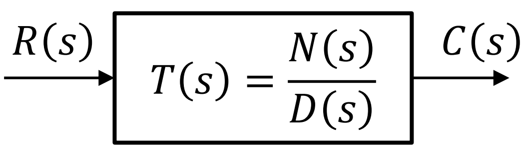


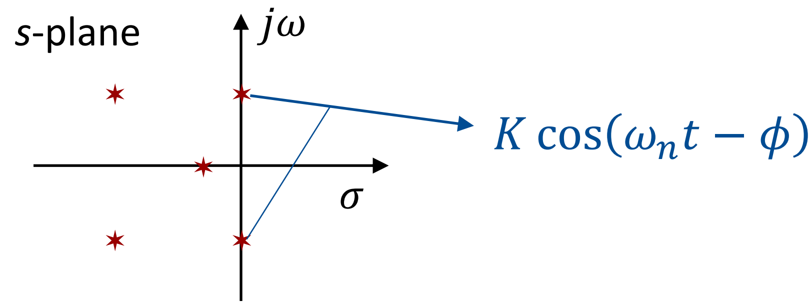




Stability Determination Using Poles: Marginally Stable System

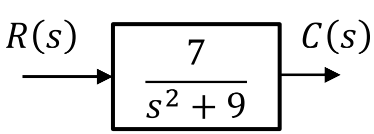
(2) If the closed-loop transfer function has **poles of multiplicity 1 only on the imaginary axis** and (possible) **poles in the left half-plane** (LHP), the system is **marginally stable**.

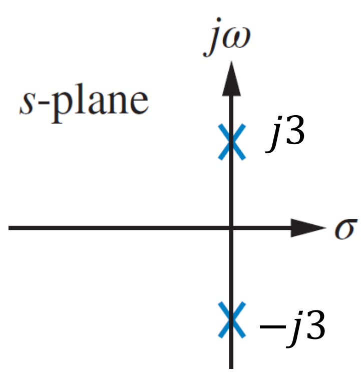




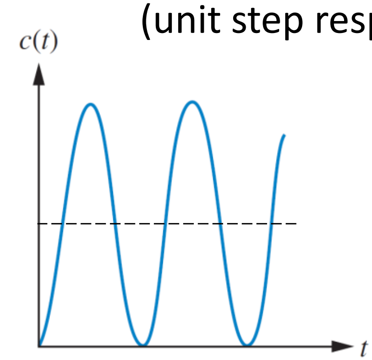
The natural response $c_n(t)$ **neither increase nor decrease** in amplitude.

Example:



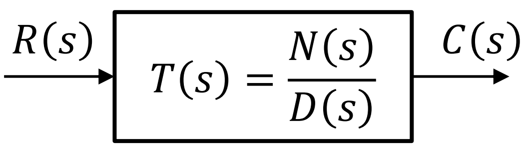


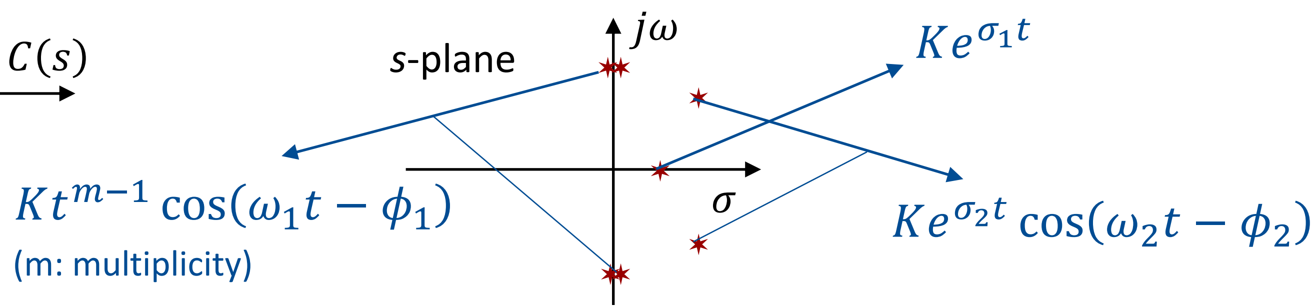
(unit step response)



Stability Determination Using Poles: Unstable System

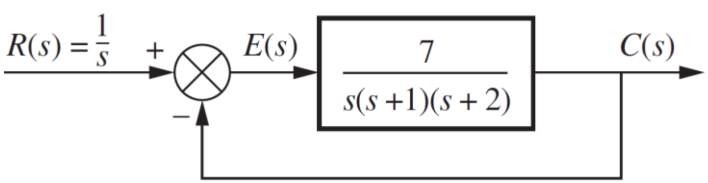
(3) If the closed-loop transfer function has **at least one pole in the right half-plane** (RHP) and/or **poles of multiplicity greater than 1 on the imaginary axis**, the system is **unstable**.

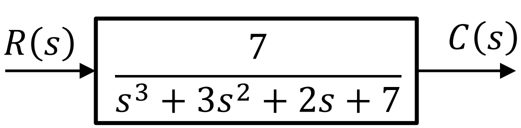


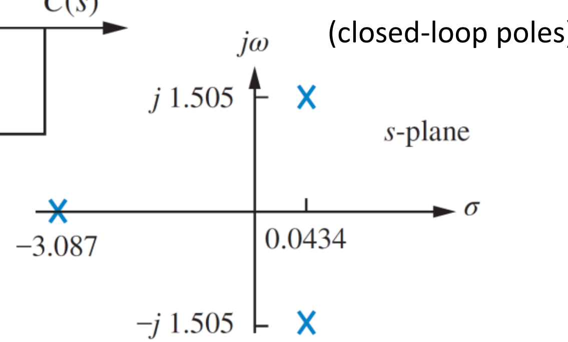


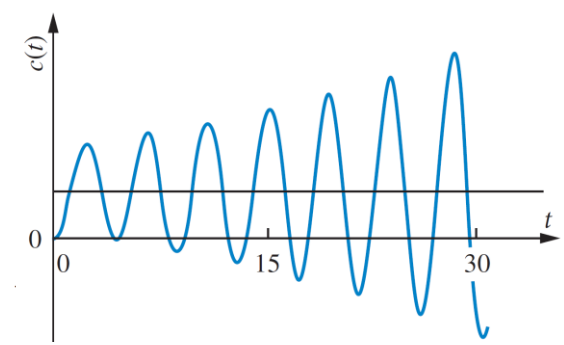
The natural response $c_n(t)$ approach **infinity** as time approaches **infinity**.

Example:



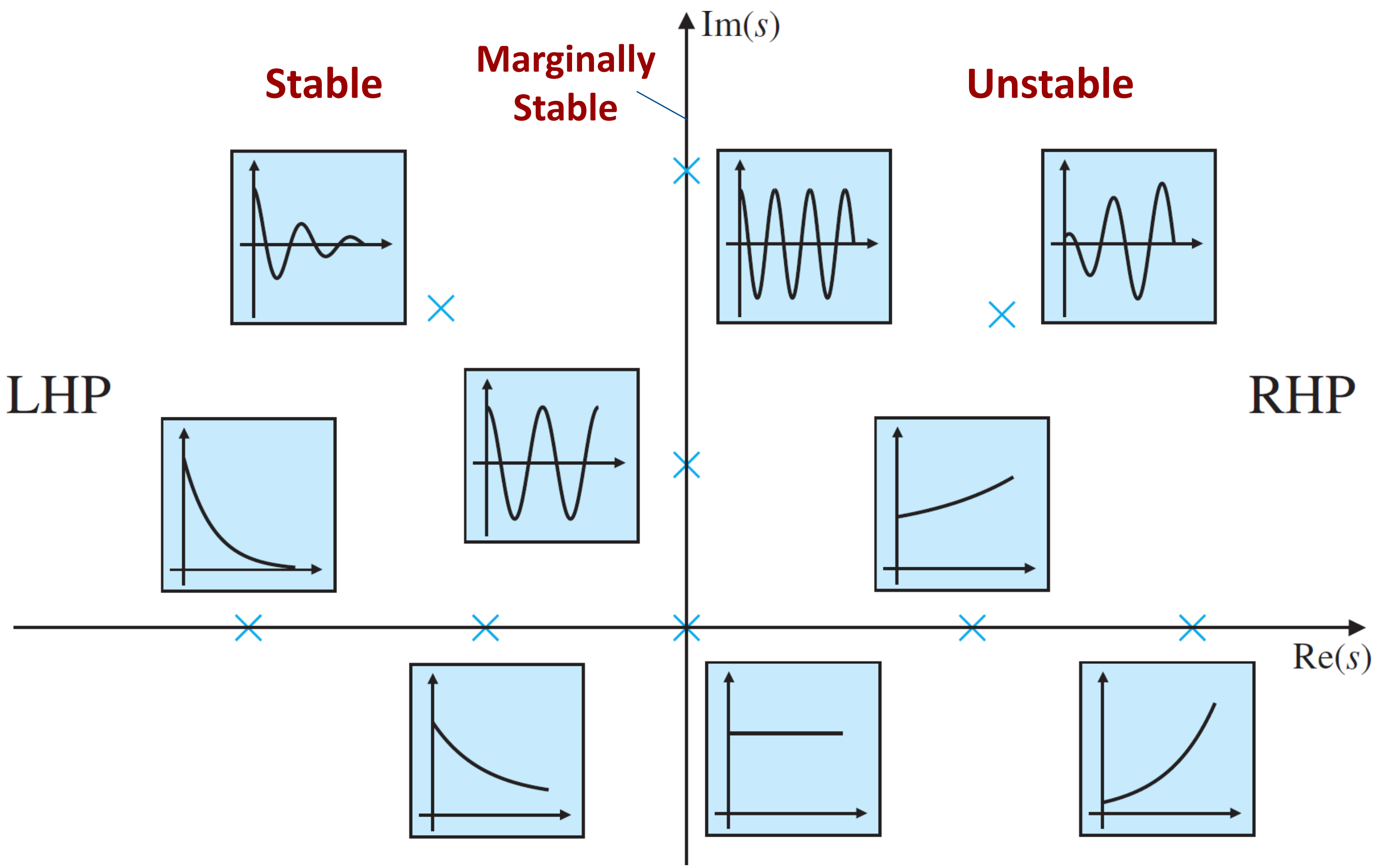






Stability Determination Using Poles: Summery

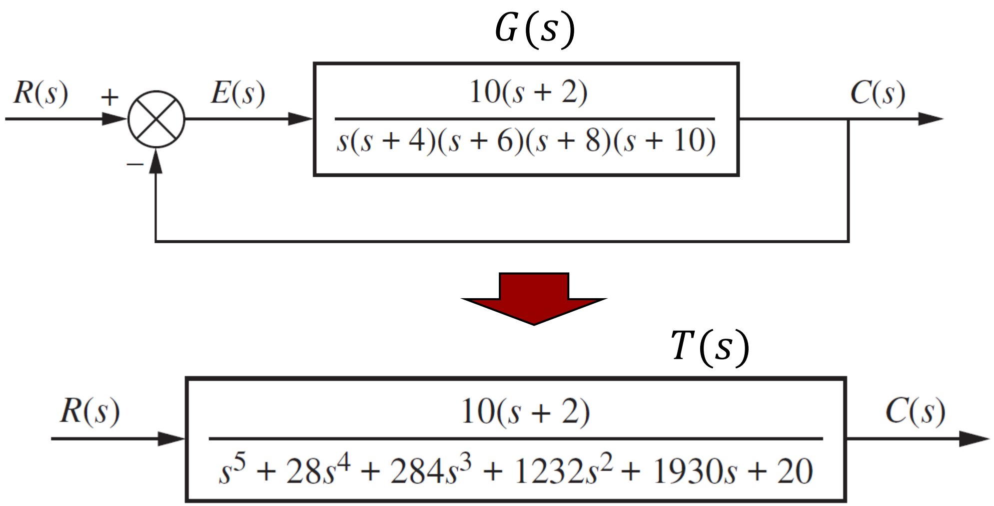
A sketch of pole locations and corresponding natural responses:



How to Determine If a System Is Stable?

It is not always simple to determine if a closed-loop system is stable.

For Example:



Although the poles of the forward transfer function $G(s)$ can be found easily, finding the **poles** of the equivalent **closed-loop system** $T(s)$ needs complicated calculations.

How to Determine If a System Is Stable?

$R(s)$ →

$$T(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

→ $C(s)$

(a) If **all signs** of the coefficients of $D(s)$ are **not the same**, the system is **unstable**.

$$D(s) = s^3 + 3s^2 - 7s + 4$$

(b) If some **powers of** s in $D(s)$ are missing, the system is either **unstable** or **marginally stable**.

$$D(s) = s^3 + 3s^2 + 4$$

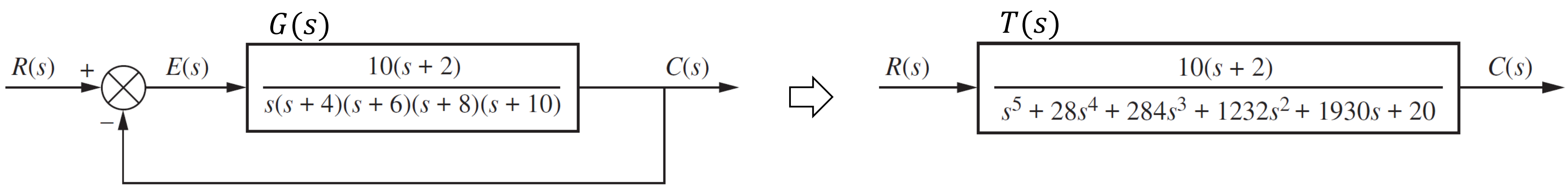
missing power: s

- If **all coefficients** of $D(s)$ are **positive** and **not missing** (these are the necessary but not sufficient condition for stability), a computer can be used to calculate the root locations of $D(s)$.

Using MATLAB and Control System Toolbox

Stability Determination Using roots & pole

MATLAB can solve for the poles of a transfer function to determine stability.



If the denominator $D(s)$ of a closed-loop transfer function is given, we can use command roots and if the transfer function $T(s)$ is given (or can be found), we can use command pole.

```
roots([1 28 284 1232 1930 20])
```

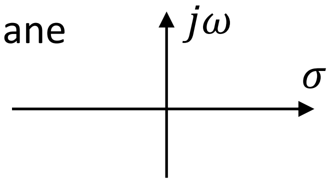
```
G = tf(10*poly([-2]), poly([0 -4 -6 -8 -10]));
% or
% G = zpk(-2,[0 -4 -6 -8 -10],10);
T = feedback(G, 1);
pole(T)
```

Routh-Hurwitz Criterion

Routh-Hurwitz Criterion

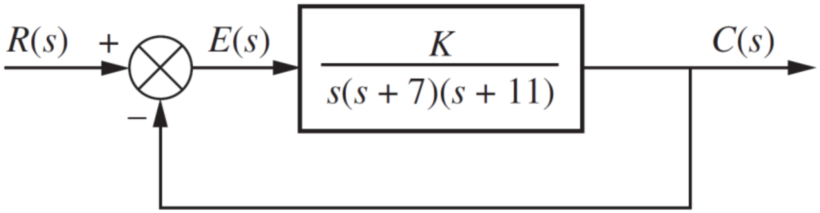
Routh-Hurwitz Criterion can determine the **number** of closed-loop system poles that are in the left half-plane (LHP), in the right half-plane (RHP), and on the $j\omega$-axis without having to solve for the roots of $D(s)$ (notice it determines how many, not where).

s-plane



- Although modern calculators can calculate the exact location of system poles, the power of the Routh-Hurwitz criterion lies in **design** rather than analysis.

  For example, Routh-Hurwitz Criterion can yield a closed-form expression for the range of the unknown parameter K to yield stability.

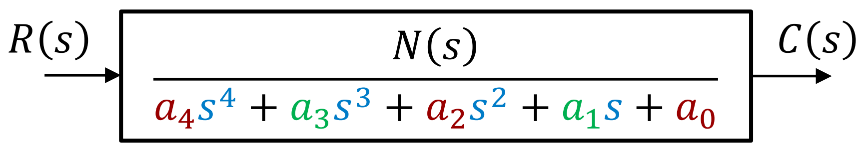


$$T(s) = \frac{K}{s^3 + 18s^2 + 77s + K}$$

Generating a Routh Table

The method has two steps: (1) **Generating** a **Routh Table**, (2) **Interpreting** the **Routh Table**.

- First create the Routh Table by labeling the rows with powers of s from the highest power of the **denominator** of the closed-loop transfer function to s^0.



s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2			
s^1			
s^0			

- **In the 1st row**, list every other coefficient of the polynomial starting with the highest power of s.
- **In the 2nd row**, list every other coefficient of the polynomial starting with the next highest power of s.

Generating a Routh Table

- Each other element is a **negative determinant** of elements in the previous two rows **divided** by the element in the first column directly above the calculated row. The left-hand column of the determinant is always the first column of the previous two rows, and the right-hand column is the elements of the column above and to the right.

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\dfrac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\dfrac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	0
s^1	$\dfrac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	0	0
s^0	$\dfrac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	0	0

Note:
For convenience, any row of the Routh Table can be **multiplied/divided** by a **positive** constant without changing the values of the rows below.

Interpreting the Routh Table

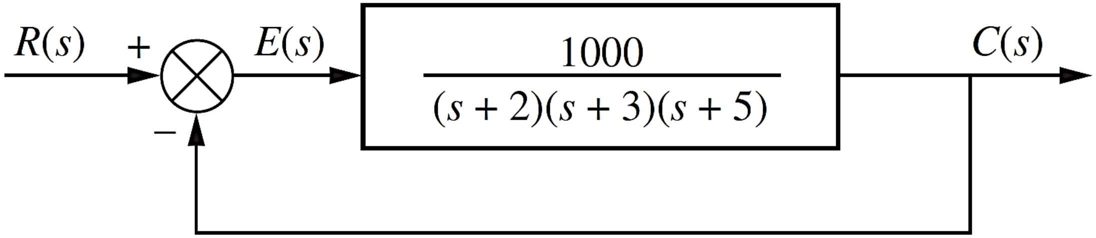
Routh-Hurwitz Criterion declares that the **number of closed-loop system poles** that are in the **right half-plane** (RHP) is equal to the number of **sign changes** in the first column.

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\dfrac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\dfrac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	
s^1	$\dfrac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$		
s^0	$\dfrac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$		

Thus, a system is **stable** if there are **no sign changes** in the first column of the Routh table.

Example

Make the Routh table for the system and consider its stability.



Answer: The system is unstable since two poles exist in the right half-plane.

Example

Make the Routh table for the closed-loop system and consider its stability.

$$T(s) = \frac{10}{s^4 + 2s^3 + 3s^2 + 4s + 5}$$

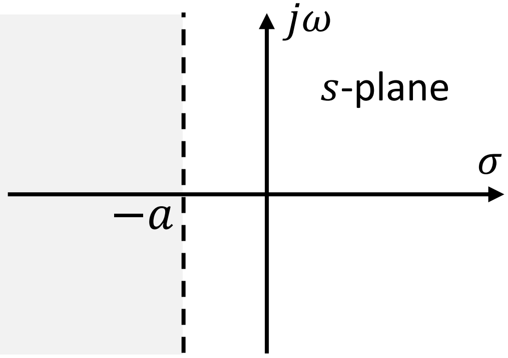
Answer: The system is unstable since two poles exist in the right half-plane.

Relative Stability Analysis

In designing a control system, it is necessary that the system has adequate **Relative Stability**. Relative stability is a measure of how close a system is to instability (or a measure of how close the poles of a system are to the $j\omega$-axis).

For examining relative stability, shift the $j\omega$-axis by substituting $s = \hat{s} - a$ into the denominator polynomial $D(s)$ of the closed-loop system and writing the polynomial in terms of $\hat{s}$.

$$D(s) \xrightarrow{s = \hat{s} - a} D(\hat{s})$$



Then, apply Routh-Hurwitz Criterion to the new polynomial $D(\hat{s})$. The number of changes of sign in the first column of the Routh Table is equal to the number of roots of the original polynomial $D(s)$ that are located to the right of the vertical line $s = -a$.

Special Cases

Special Case 1: Zero Only in the First Column (Method 1)

If the first element of a row is zero, division by zero would be required to form the next row. To avoid this, two methods can be used.

Method 1:
In this method, the zero term is replaced by a **very small positive number** ϵ and the rest of the elements in the table are computed in terms of ϵ. Then, the signs of the elements in the first column can be determined.

Example:

$$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$$

s^5	1	3	5
s^4	2	6	3
s^3	~~0~~ ϵ (+)	$\frac{7}{2}$	0
s^2	$\frac{6\epsilon - 7}{\epsilon}$ (-)	3	0
s^1	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$ (+)	0	0
s^0	3	0	0

Unstable: Two poles in the RHP

Special Case 1: Zero Only in the First Column (Method 2)

Method 2:
In this method, the original polynomial $D(s)$ is replaced by a polynomial $D(\hat{s})$ that has the reciprocal roots of the original polynomial $D(s)$, then, the Routh Table for the new polynomial $D(\hat{s})$ will possibly not have a zero in the first column.
Note: Taking the reciprocal of a root value does not move it to another half plane.

$$D(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0 \xrightarrow{s=1/\hat{s}} \left(\frac{1}{\hat{s}}\right)^n \underbrace{[1 + a_{n-1}\hat{s} + \cdots + a_1\hat{s}^{n-1} + a_0\hat{s}^n]}_{} = 0$$

Thus, the polynomial with reciprocal roots is a polynomial with the coefficients written in reverse order.

Example:

$$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$$

$$D(\hat{s}) = 3\hat{s}^5 + 5\hat{s}^4 + 6\hat{s}^3 + 3\hat{s}^2 + 2\hat{s} + 1$$

$\hat{s}^5$	3	6	2
$\hat{s}^4$	5	3	1
$\hat{s}^3$	4.2	1.4	
$\hat{s}^2$	1.33	1	
$\hat{s}^1$	−1.75		
$\hat{s}^0$	1		

Unstable: Two poles in the RHP

Special Case 2: Entire Row is Zero

If all the coefficients in any derived row are zero, the evaluation of the rest of the array can be continued by forming an **auxiliary polynomial** $P(s)$ with the coefficients of the **last non-zero row** (starting with the power of s in the label column and continue by skipping every other power of s) and by using the coefficients of the **derivative** of this auxiliary polynomial $P(s)$ in the next row.

Example:

$$T(s) = \frac{10}{s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56}$$

$$P(s) = s^4 + 6s^2 + 8 \quad \leftarrow \; s^4$$

$$\frac{dP(s)}{ds} = 4s^3 + 12s + 0 \quad \rightarrow \; s^3$$

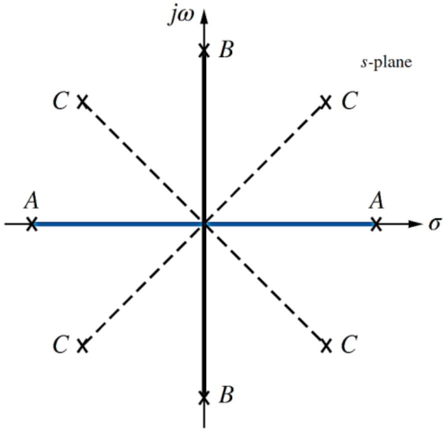
s^5	1	6	8
s^4	~~7~~ → 1	~~42~~ → 6	~~56~~ → 8
s^3	~~0~~ → ~~4~~ → 1	~~0~~ → ~~12~~ → 3	~~0~~ → ~~0~~ → 0
s^2	3	8	0
s^1	$\frac{1}{3}$	0	0
s^0	8	0	0

Dividing the row by a positive constant only for simplification.

Special Case 2: Important Comments

- An entire row of zeros will appear only in an **odd-numbered** row.
- An entire row of zeros will appear when an **Even Polynomial** is **a factor of the original polynomial** (an even polynomial has only **even powers** of s). This even polynomial is actually the **Auxiliary Polynomial** $P(s)$. Thus, $P(s)$ is always a factor of the denominator $D(s)$, i.e., $D(s) = P(s)Q(s)$.

  In the previous example: $s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56 = (s^4 + 6s^2 + 8)(s + 7)$
- Even polynomials only have roots that are **symmetrical about the origin**, i.e., each or combination of (*A*) symmetrical and real, (*B*) symmetrical and imaginary, or (*C*) quadrantal.
- Since imaginary roots are symmetric about the origin, if we do not have a row of zeros, we cannot have imaginary roots (on $j\omega$-axis). If we have a row of zeros, we may have imaginary roots (on $j\omega$-axis).



Special Case 2: Important Comments (count.)

- The number of sign changes in the Routh table from the auxiliary (or even) polynomial's row down to the end equals the number of RHP roots of the auxiliary polynomial $P(s)$. Having accounted for the roots in the RHP and LHP, the remaining roots must be on the $j\omega$-axis. If there is no sign change, all the roots of $P(s)$ will be on the $j\omega$-axis.

- The number of sign changes in the Routh table from the beginning of the table to the row containing the auxiliary polynomial $P(s)$ equals the number of RHP roots of the **other factor** $Q(s)$ of the original polynomial $D(s)$.

$$D(s) = P(s)Q(s)$$

s^5	1	6	8
s^4	~~7~~ 1	~~42~~ 6	~~56~~ 8
s^3	~~0~~ ~~4~~ 1	~~0~~ ~~12~~ 3	~~0~~ ~~0~~ 0
s^2	3	8	0
s^1	$\frac{1}{3}$	0	0
s^0	8	0	0

Corresponding to RHP roots of $Q(s)$ (rows s^5–s^4)

Corresponding to RHP roots of $P(s)$ (rows s^4–s^0)

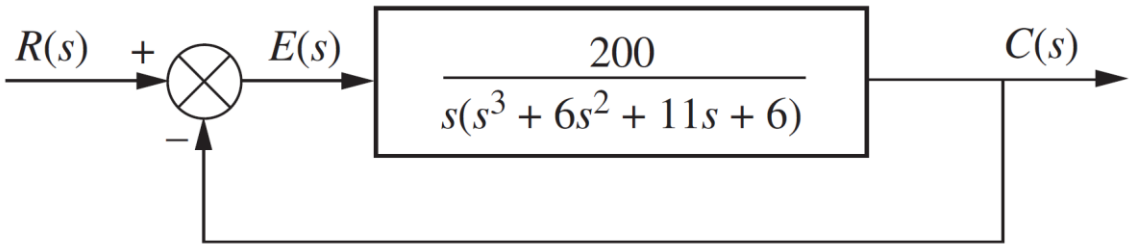
s^4 row → $P(s)$

(Roost of $s^4 + 6s^2 + 8$: $\pm\sqrt{2}j, \pm 2j$)

$\Rightarrow$ This system is **marginally stable**.

Example

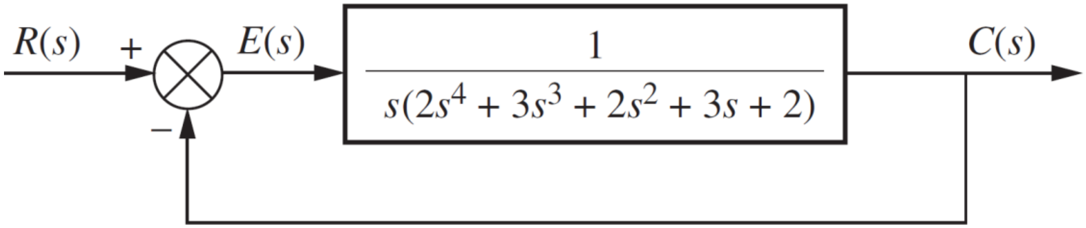
Find the number of poles in the left half-plane, the right half-plane, and on the $j\omega$-axis for the system.



Answer: The system has two poles in the right half-plane, two poles in the left half-plane, and no pole on the $j\omega$-axis. Thus, the system is unstable.

Example

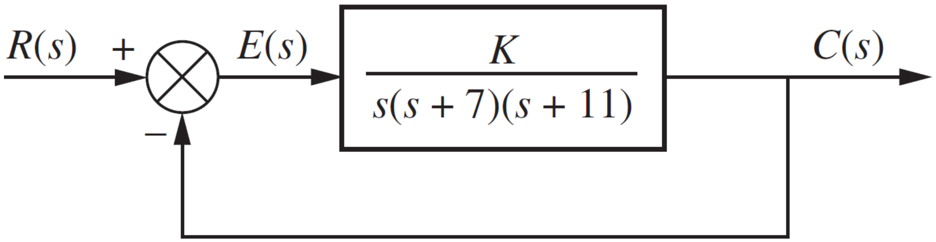
Find the number of poles in the left half-plane, the right half-plane, and on the $j\omega$-axis for the system



Answer: The system has two poles in the right half-plane, three poles in the left half-plane, and no pole on the $j\omega$-axis. Thus, the system is unstable.

Example

Find the range of gain K for the system that will cause the system to be stable, unstable, and marginally stable. Assume $K > 0$.
Find the frequency of oscillation for the marginally stable case.



Answer: If $K < 1386$, the system is stable; if $K > 1386$, the system is unstable; if $K = 1386$, the system is marginally stable, and the frequency of oscillation is $\sqrt{77}$.

Example

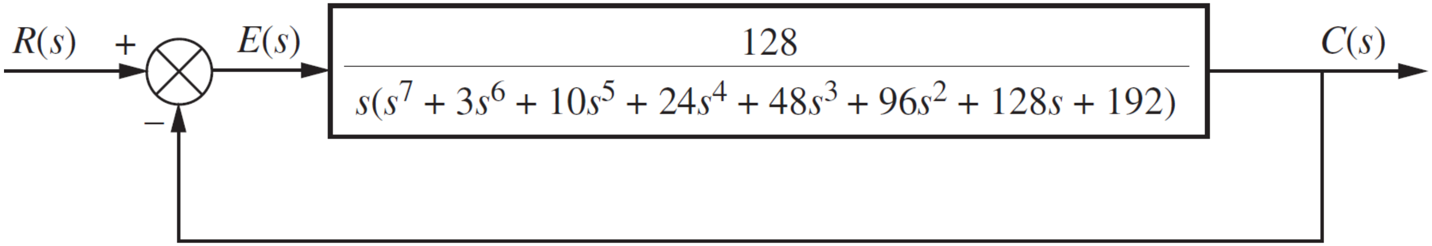
For the transfer function, tell how many poles are in the right half-plane, in the left half-plane, and on the $j\omega$-axis.

$$T(s) = \frac{20}{s^8 + s^7 + 12s^6 + 22s^5 + 39s^4 + 59s^3 + 48s^2 + 38s + 20}$$

Answer: The system has two poles in the right half-plane, two poles in the left half-plane, and four poles on the $j\omega$-axis. Thus, the system is unstable.

Example

Find the number of poles in the left half-plane, the right half-plane, and on the $j\omega$-axis for the system. Draw conclusions about the stability of the closed-loop system.



Answer: The system has two poles in the right half-plane, four poles in the left half-plane, and two poles on the $j\omega$-axis. Thus, the system is unstable.