Ch8: Steady-State Errors

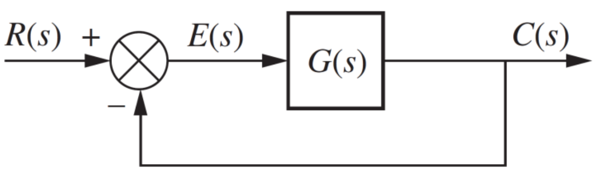
Contents:

Definition

Steady-State Error Definition

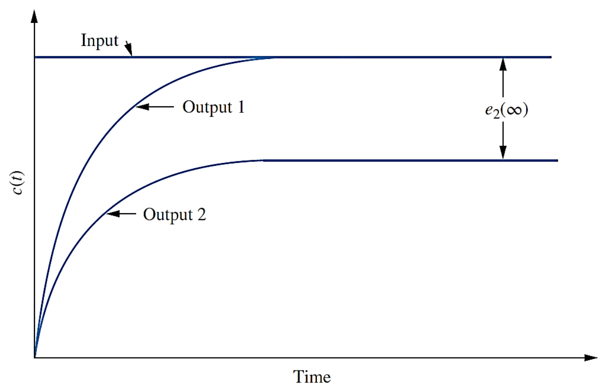
Steady-State Error is the difference between the input $R(s)$ and the output $C(s)$ as $t \to \infty$ (i.e., $E(s) = R(s) - C(s)$). Note that this concept is applicable only to **stable** systems in which the **input and output have the same units**.

- **Test inputs** $R(s)$ used for steady-state error analysis and design are Step, Ramp, and Parabola.

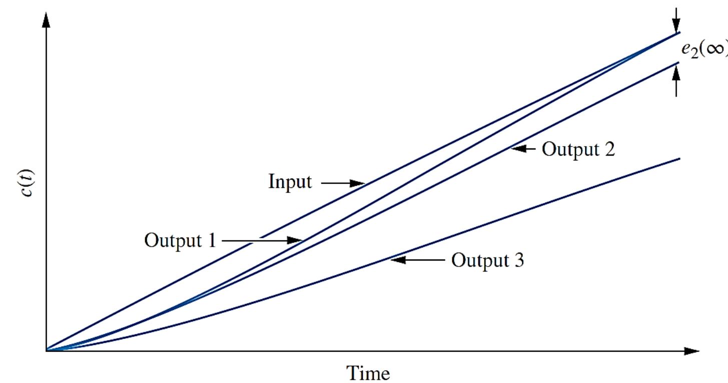


Examples:

Step Input:
Output 1 has zero steady-state error,
Output 2 has a finite steady-state error, $e_2(\infty)$.



Ramp Input:
Output 1 has zero steady-state error,
Output 2 has a finite steady-state error, $e_2(\infty)$,
Output 3 has an infinite steady-state error.



Note: Errors are measured **vertically** between the input and output as $t \to \infty$.

Sources of Steady-State Error

Steady-state errors in control systems arise from two sources:

(1) Imperfections and nonlinearities in the system components, such as backlash in gears, static friction, or motor dead zone (when a motor will not move unless the input voltage exceeds a threshold).

(2) The configuration (type) of open-loop transfer function of the system $G(s)$ itself and the type of applied input $R(s)$. This kind of error will be investigated in this chapter.

Any physical control system inherently suffers steady-state error in response to **certain types of inputs**. A system may have no steady-state error to a step input, but the same system may exhibit nonzero steady-state error to a ramp input. The only way we may be able to eliminate this error is to **modify the system structure**.

Steady-State Error for Unity Feedback Systems

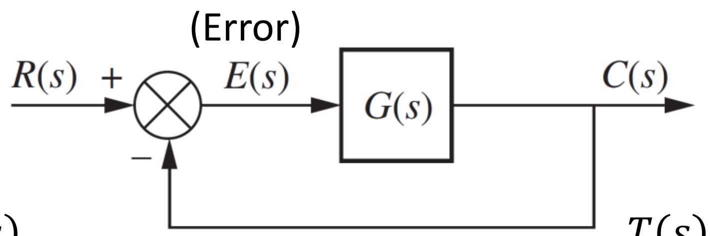
Steady-State Error in Terms of $G(s)$

Consider the feedback control system which has **unity negative feedback** (i.e., $H(s) = 1$).

$$\left.\begin{aligned} E(s) &= R(s) - C(s) \\ C(s) &= E(s)G(s) \end{aligned}\right\} \quad E(s) = \frac{R(s)}{1 + G(s)}$$

(or)

$$E(s) = R(s) - C(s) = R(s) - R(s)T(s) = \frac{R(s)}{1 + G(s)}$$



$$T(s) = \frac{G(s)}{1 + G(s)}$$

closed-loop system

Note: You should always first make sure the closed-loop system $T(s)$ is **stable**, using a method like the Routh-Hurwitz criterion.

Since we are interested in the final value of the error, $e(\infty)$, the **final value theorem*** is used:

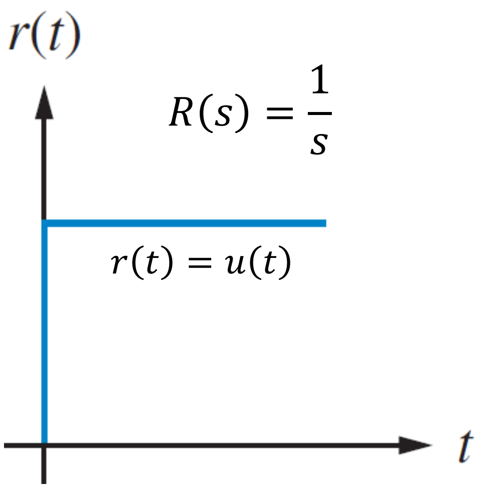
$$e(\infty) = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) \quad \Rightarrow \quad e(\infty) = \lim_{s\to 0} \frac{sR(s)}{1 + G(s)}$$

* **Note**: The final value theorem is valid only if $E(s)$ has poles only in the left half-plane and, at most, one pole at the origin. For infinite steady-state errors, the final value theorem is also valid if $E(s)$ has more than one pole at the origin.

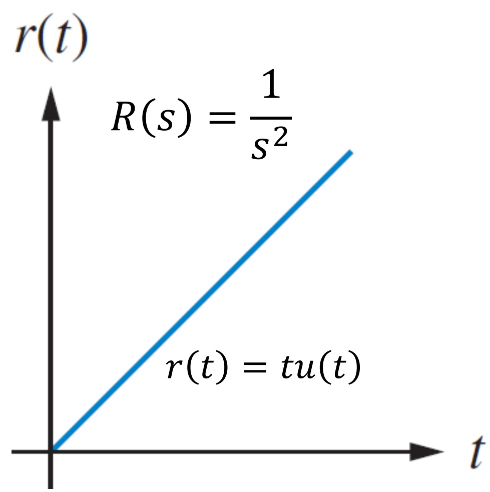
Steady-State Error for Three Test Inputs

Now, the relationships between the open-loop system, $G(s)$, and the nature of the steady-state error, $e(\infty)$, is studied for the three test inputs.

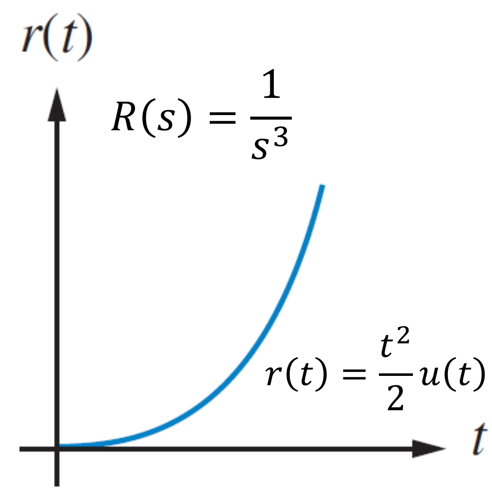
1. Step



2. Ramp



3. Parabola



1. Step Input

$$R(s) = \frac{1}{s} \quad \rightarrow \quad e_{\text{step}}(\infty) = \lim_{s\to 0} \frac{s(1/s)}{1+G(s)} = \frac{1}{1+\lim_{s\to 0} G(s)} = \frac{1}{1+K_p}$$

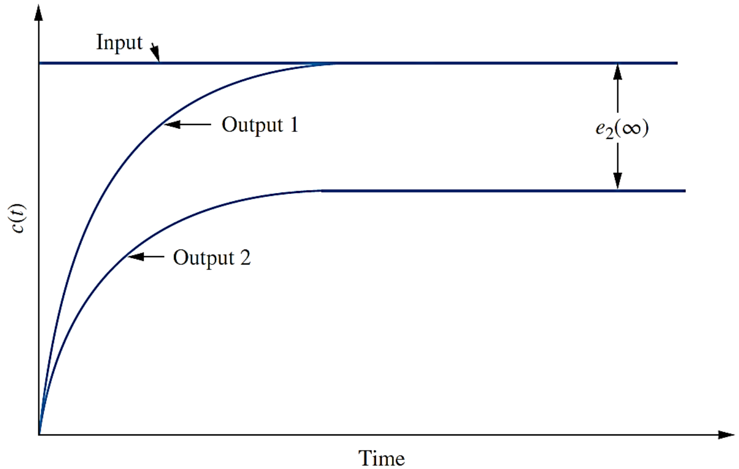
$K_p = \lim_{s\to 0} G(s)$ is called **Static Position Error Constant**.

$G(0)$ is called DC gain of the forward transfer function.

- To have zero steady-state error (Output 1), we need $K_p = \lim_{s\to 0} G(s) = \infty$. Hence, $G(s)$ must have the following form:

$$G(s) = \frac{(s+z_1)(s+z_2)\cdots}{s^N(s+p_1)(s+p_2)\cdots} \qquad N \geq 1$$

i.e., at least one pole must be at the origin.



- If $N = 0$, then $K_p = z_1 z_2 \ldots / p_1 p_2 \ldots$ and there is a constant steady-state error (Output 2).

2. Ramp Input

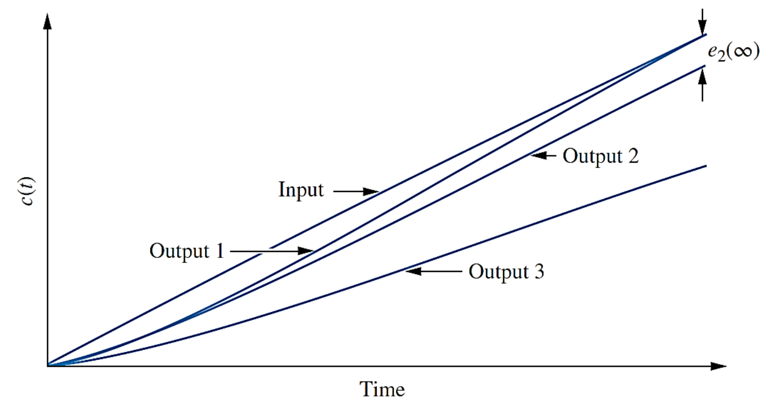
$$R(s) = \frac{1}{s^2} \longrightarrow e_{\text{ramp}}(\infty) = \lim_{s\to 0} \frac{s(1/s^2)}{1+G(s)} = \lim_{s\to 0} \frac{1}{s + sG(s)} = \frac{1}{\lim_{s\to 0} sG(s)} = \frac{1}{K_v}$$

$K_v = \lim_{s\to 0} sG(s)$ is called **Static Velocity Error Constant**.

- To have zero steady-state error (Output 1), we need $K_v = \lim_{s\to 0} sG(s) = \infty$. Hence, $G(s)$ must have the following form:

$$G(s) = \frac{(s+z_1)(s+z_2)\cdots}{s^N(s+p_1)(s+p_2)\cdots} \qquad N \geq 2$$

i.e., at least two pole must be at the origin.



- If $N = 1$, then $K_v = z_1 z_2 \ldots / p_1 p_2 \ldots$ and there is a constant steady-state error (Output 2).
- If $N = 0$, then $K_v = 0$ and the steady-state error would be infinite (Output 3).

3. Parabolic Input

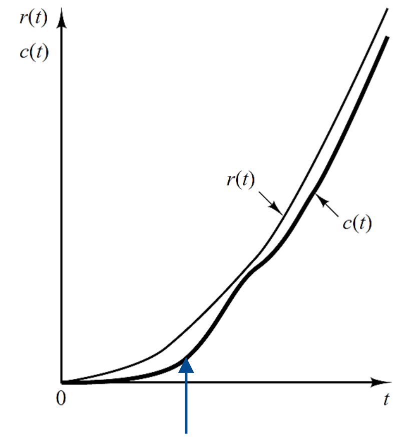
$$R(s)=\frac{1}{s^3} \longrightarrow e_{\text{parabol}}(\infty)=\lim_{s\to 0}\frac{s(1/s^3)}{1+G(s)}=\lim_{s\to 0}\frac{1}{s^2+s^2G(s)}=\frac{1}{\lim_{s\to 0}s^2G(s)}=\frac{1}{K_a}$$

$K_a=\lim_{s\to 0}s^2G(s)$ is called **Static Acceleration Error Constant**.

- To have zero steady-state error, we need $K_a=\lim_{s\to 0}s^2G(s)=\infty$. Hence, $G(s)$ must have the following form:

$$G(s)=\frac{(s+z_1)(s+z_2)\cdots}{s^N(s+p_1)(s+p_2)\cdots} \qquad N\geq 3$$

i.e., at least three pole must be at the origin.

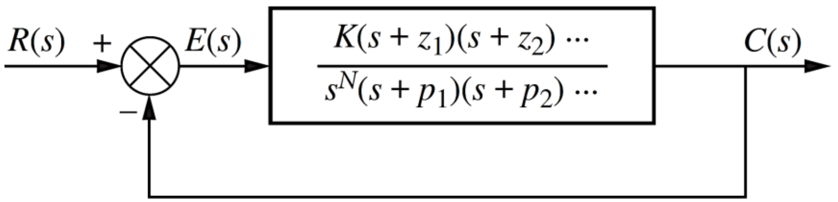


- If $N=2$, then $K_a=z_1z_2\ldots/p_1p_2\ldots$ and there is a constant steady-state error.
- If $N=1$ or $N=0$, then $K_a=0$ and the steady-state error would be infinite.

System Type and Steady-State Error

Since static error constants (position, velocity, and acceleration) and consequently steady-state errors depend upon the form of $G(s)$, **System Type** is defined to be the value of N in the denominator of $G(s)$ in the unity feedback system.

A system $G(s)$ is called **Type 0**, **Type 1**, **Type 2**,..., if $N = 0$, $N = 1$, $N = 2$,..., respectively.

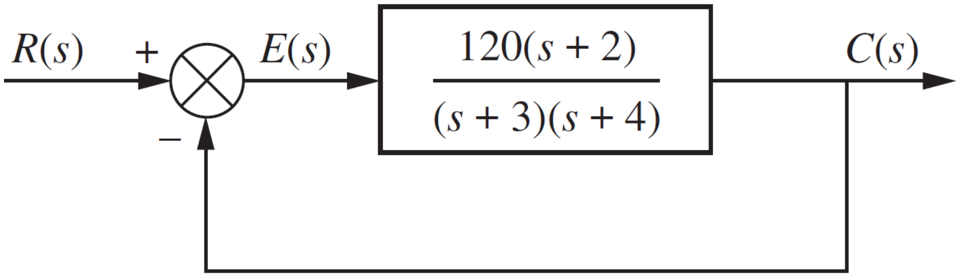


- This classification is different from that of the **order of a system**.
- As the type number is increased, **accuracy** is improved; however, **stability** is aggravated.

	Type 0		Type 1		Type 2	
Input	Static error constant	Error	Static error constant	Error	Static error constant	Error
Step, $u(t)$	$K_p = \text{Constant}$	$\dfrac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $tu(t)$	$K_v = 0$	∞	$K_v = \text{Constant}$	$\dfrac{1}{K_v}$	$K_v = \infty$	0
Parabola, $\frac{1}{2}t^2u(t)$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a = \text{Constant}$	$\dfrac{1}{K_a}$

Example

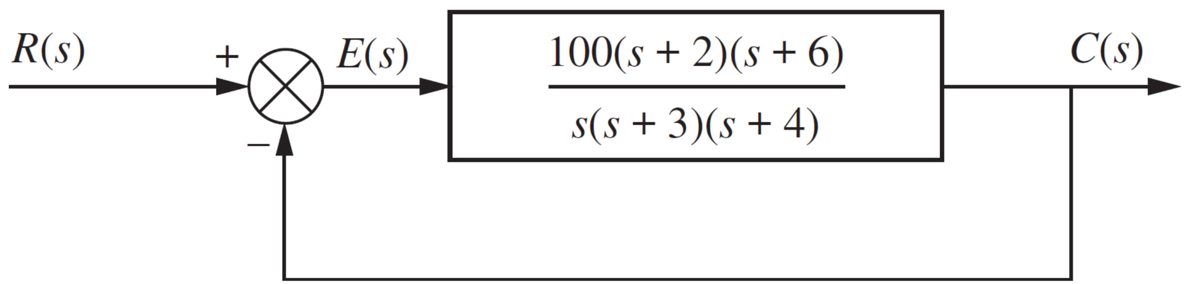
Find the steady-state errors for inputs of $5u(t)$, $5t$, and $5t^2$ to the system.



Answer: $e_{\text{step}}(\infty) = \frac{5}{21}, e_{\text{ramp}}(\infty) = \infty, e_{\text{parabola}}(\infty) = \infty$

Example

Find the steady-state errors for inputs of $5u(t)$, $5t$, and $5t^2$ to the system.



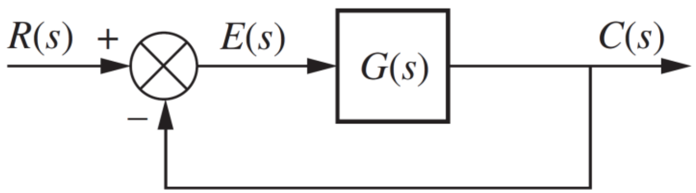
Answer: $e_{\text{step}}(\infty) = 0, e_{\text{ramp}}(\infty) = \frac{1}{20}, e_{\text{parabola}}(\infty) = \infty$

Steady-State Error Specifications

- Similar to the parameters represent performance specifications for the **transient response** (e.g., damping ratio ζ, settling time T_s, peak time T_p, and maximum overshoot M_p), Static Error Constants (e.g., position constant K_p, velocity constant K_v, and acceleration constant K_a) represent **steady-state error** performance specifications.

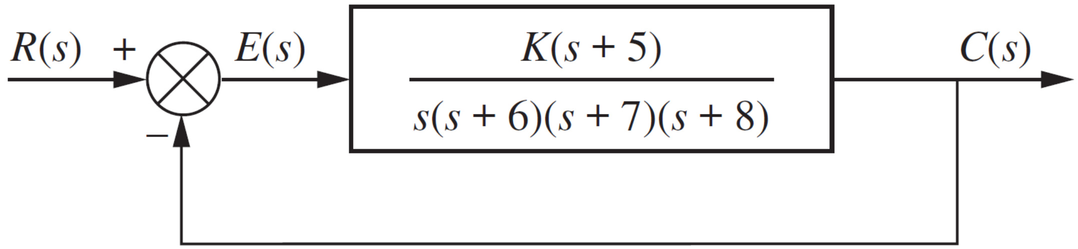
- The finite steady-state errors decreases as the static error constants increases.

- A wealth of information is contained within the specification of a static error constant. For example, if a control system has the specification $K_v = 1000$, we can draw several conclusions:
  1. The system is **stable**.
  2. The system is of **Type 1**.
  3. A **ramp input** is the test signal.
  4. The steady-state error is $1/K_v$.



Example

Find the value of K so that there is 10% error in the steady state for ramp input.

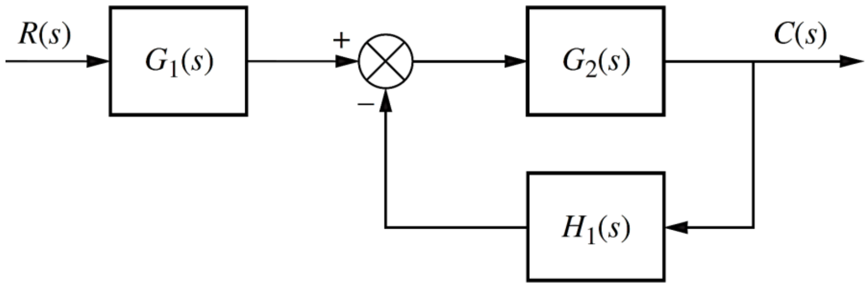


Answer: $K = 672$

Steady-State Error for Nonunity Feedback Systems

Nonunity Feedback Systems

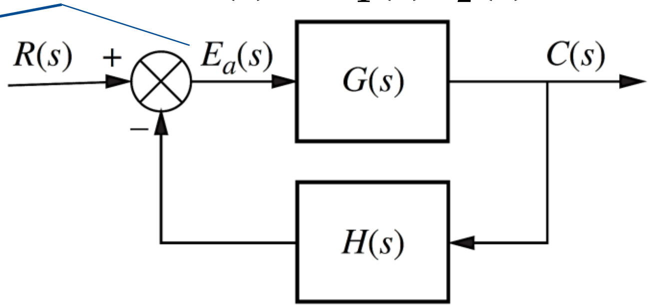
Control systems often do not have unity feedback. A general feedback system including the input transducer $G_1(s)$, controller and plant $G_2(s)$, and feedback $H_1(s)$, is shown.



Pushing the input transducer $G_1(s)$ to the right past the summing junction yields another form of a general nonunity feedback system.

$G(s) = G_1(s)G_2(s)$

$E_a(s)$ is called **Actuating Signal** (not Error!):



$$\left.\begin{aligned} E_a(s) &= R(s) - C(s)H(s) \\ E_a(s)G(s) &= C(s) \end{aligned}\right\} \quad E_a(s) = \frac{R(s)}{1 + G(s)H(s)}$$

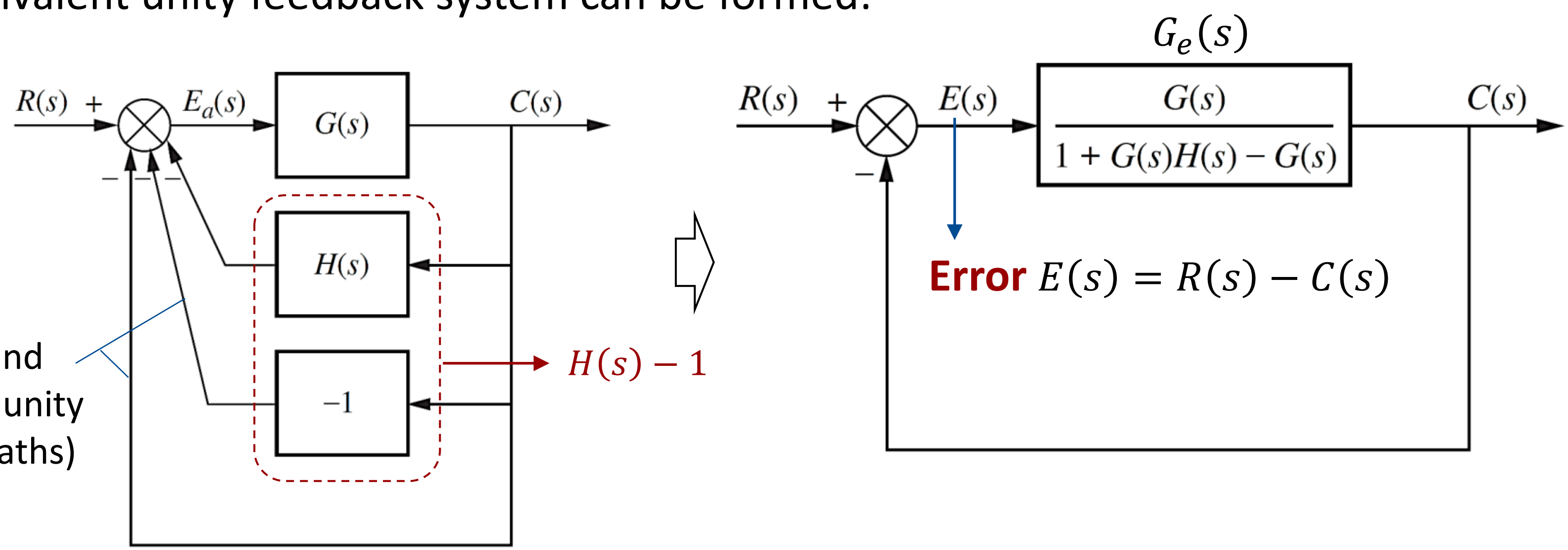
$H(s) = H_1(s)/G_1(s)$

Nonunity Feedback Systems

For Nonunity Feedback Systems we can find <u>two steady state specifications</u>:

1. **Steady-State Actuating Signal** $e_a(\infty)$: $$e_a(\infty) = \lim_{s\to 0} sE_a(s) = \lim_{s\to 0} \frac{sR(s)}{1 + G(s)H(s)}$$

2. **Steady-State Error** $e(\infty)$ (only if input and output units are the same):
   An equivalent unity feedback system can be formed:

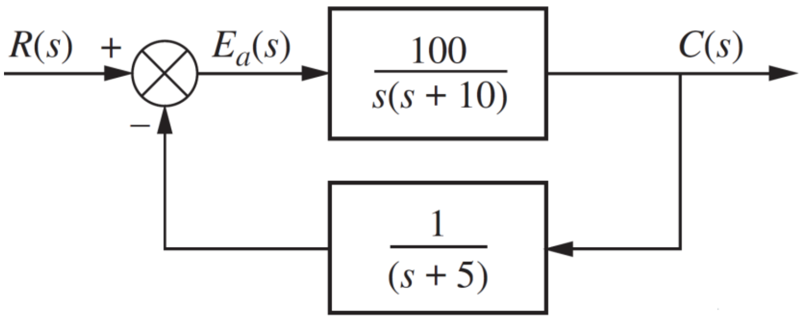


(adding and subtracting unity feedback paths)

After converting a nonunity feedback system to an equivalent unity feedback system, we can use previous methods for $G_e(s)$ for finding the Steady-State Error $e(\infty)$.

Example

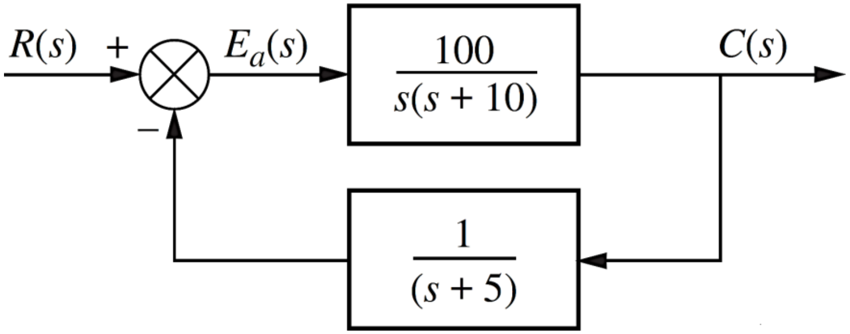
For the following system, find the system type, the appropriate error constant associated with the system type, and the steady-state error for a unit step input. Assume input and output units are the same.



Answer: Type 0, $K_p = -5/4,\ e(\infty) = -4$

Example

Find the steady-state actuating signal for the system shown for a unit step input. Repeat for a unit ramp input.



Answer: $e_{a,\mathrm{step}}(\infty) = 0, \; e_{a,\mathrm{ramp}}(\infty) = \dfrac{1}{2}$

Sensitivity

Sensitivity

The degree to which changes in system parameters affect system transfer functions, and hence performance, is called **Sensitivity**. Ideally, parameter changes due to heat or other causes should not appreciably affect a system's performance. A system with zero sensitivity (i.e., changes in the system parameters have no effect on the transfer function) is ideal. The greater the sensitivity, the less desirable the effect of a parameter change.

- Sensitivity of function F to changes in parameter P:

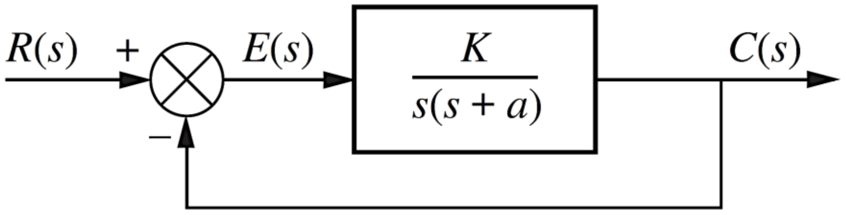
$$S_P^F = S_{F:P} = \lim_{\Delta P \to 0} \frac{\text{Fractional change in the function } (F)}{\text{Fractional change in the parameter } (P)} = \lim_{\Delta P \to 0} \frac{\Delta F / F}{\Delta P / P} = \lim_{\Delta P \to 0} \frac{P}{F} \frac{\Delta F}{\Delta P}$$

$$S_P^F = S_{F:P} = \frac{P}{F} \frac{\delta F}{\delta P}$$

- In general, feedback reduces the sensitivity to parameter changes.

Example

Given the following system, (a) calculate the sensitivity of the closed-loop transfer function to changes in the parameter a. (b) How would you reduce the sensitivity?

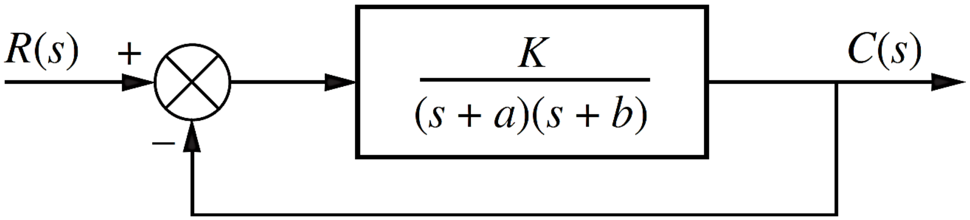


Answer:

(a) $S_{T:a} = \dfrac{-as}{s^2 + as + K}$ (b) By increasing K.

Example

Find the sensitivity of the steady-state error to changes in parameter K and parameter a for the following system with a step input.



Answer: $S_{e:K} = \dfrac{-K}{ab + K}$, $S_{e:a} = \dfrac{K}{ab + K}$

Using MATLAB and Control System Toolbox

Finding Static Error Constants Using dcgain

```
s = tf('s');
G = 500*(s+2)*(s+4)*(s+5)*(s+6)*(s+7)/...
(s^2*(s+8)*(s+10)*(s+12));

% Check Stability
T = feedback(G,1);
poles = pole(T)

% Step Input
Kp = dcgain(G);
ess_step = 1/(1+Kp)

% Ramp Input
Kv = dcgain(s*G);
ess_ramp = 1/Kv

% Parabolic Input
Ka = dcgain(s^2*G);
ess_parabolic = 1/Ka
```