# Ch9: Root Locus Techniques

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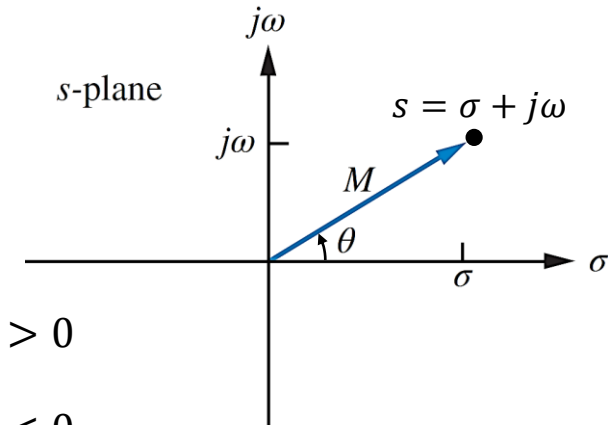
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# Introduction

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# Vector Representation of Complex Numbers

Any complex number $s = \sigma + j\omega$ in $s$-plane can be graphically represented by a **vector**. It also can be described in polar form with magnitude $M$ and angle $\theta$, as $M\angle\theta$.

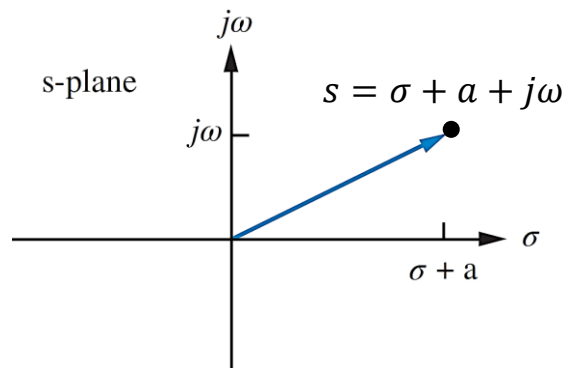$$s = \sigma + j\omega = M(\cos\theta + j\sin\theta) = M\angle\theta$$

$$M = |(\sigma + j\omega)| = \sqrt{\sigma^2 + \omega^2}, \quad \theta = \angle(\sigma + j\omega) = \begin{cases} \tan^{-1}\dfrac{\omega}{\sigma} & \sigma > 0 \\ \tan^{-1}\dfrac{\omega}{\sigma} + \pi & \sigma < 0 \end{cases}$$
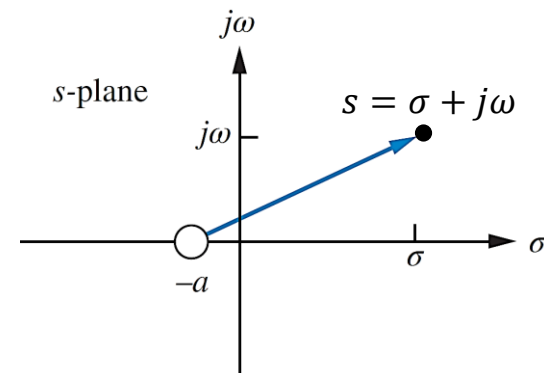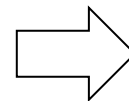
$F(s) = s + a$ (where $a$ is either real or complex) is a complex number which is calculated by substituting the complex number $s = \sigma + j\omega$ into the complex function $F(s)$.
$F(s)$ also can be represented by a vector drawn from $-a$ to the point $s$.

For example,
for real $a$:

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# Evaluation of a Complex Function via Vectors

Consider a complex function (like a transfer function) as

$$F(s) = \frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} = \frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)}$$
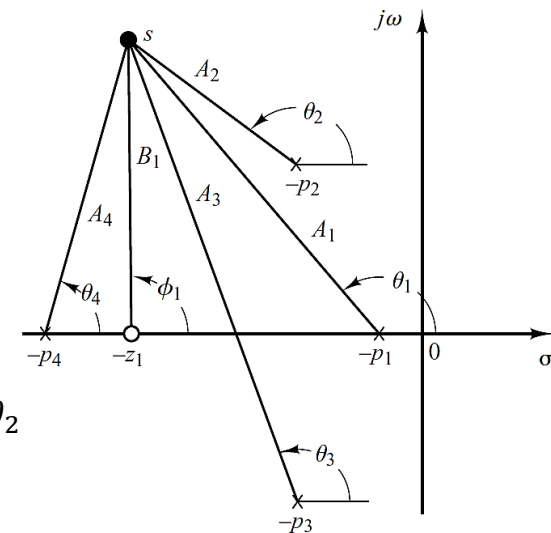
(Symbol $\Pi$ means product)

Since each complex factor in the numerator and denominator can be thought of as a vector, the magnitude $M$ and angle $\theta$ of $F(s)$ at any point $s$ are

$$M = |F(s)| = \frac{\prod_{i=1}^{m}|(s+z_i)|}{\prod_{j=1}^{n}|(s+p_j)|} = \frac{\prod \text{zero lengths}}{\prod \text{Pole lengths}}$$

$$\theta = \angle F(s) = \sum_{i=1}^{m} \angle(s+z_i) - \sum_{j=1}^{n} \angle(s+p_j)$$

$$= \sum \text{zero angles} - \sum \text{ploe angles}$$

$$M = \frac{B_1}{A_1 A_2 A_3 A_4}$$

$$\theta = \phi_1 - (\theta_1 + \theta_2)$$

- $|(s+a_i)|$ is the magnitude and $\angle(s+a_i)$ is the angle (measured from the **positive** extension of the real axis $\sigma$) of the vector drawn from the zero $-z_i$ or pole $-p_j$ of $F(s)$ to the point $s$.

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# Example

Using $s$-plane, find $F(s)$ at the point $s = -3 + j4$.

$$F(s) = \frac{(s + 1)}{s(s + 2)}$$

**Answer**:    $M\angle\theta = 0.217\angle - 114.3°$

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# Infinite Poles and Zeros

A function $G(s)$ can have **poles and zeros at infinity**.

$$G(s) = \frac{(s + z_1)(s + z_2)\cdots(s + z_m)}{(s + p_1)(s + p_2)\cdots(s + p_n)}$$

- If the function $G(s)$ approaches **infinity** as $s$ approaches **infinity**, then the function has a **pole at infinity**. For example,

$$G(s) = s \qquad \text{has a pole at infinity}$$

- If the function $G(s)$ approaches **zero** as $s$ approaches **infinity**, then the function has a **zero at infinity**. For example,

$$G(s) = 1/s \qquad \text{has a zero at infinity}$$

❖ Every function $G(s)$ has always an **equal number of poles and zeros** if we include the **infinite poles and zeros** as well as the **finite poles and zeros**. For example,

$$G(s) = \frac{1}{s(s+1)(s+2)} \qquad \text{has 3 finite poles and 3 infinite zeros.}$$

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# Defining Root Locus

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# Defining Root Locus

- The system's <u>transient response</u> and <u>stability</u> are dependent upon the poles of the closed-loop transfer function $T(s)$.
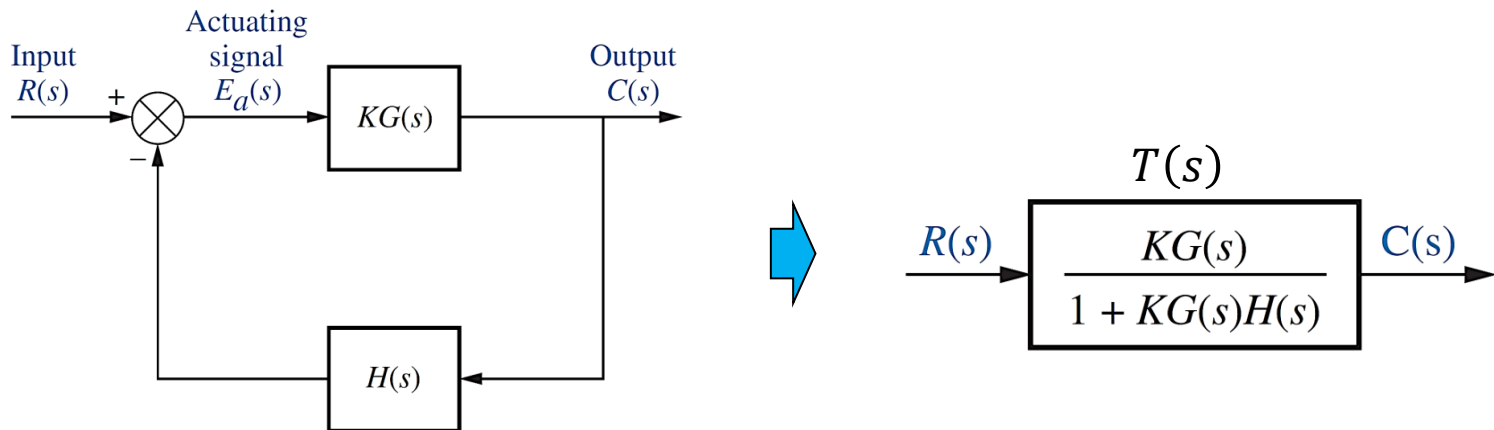


- Typically, the poles of the open-loop transfer function are easily found, but the poles of the closed-loop transfer function are more difficult to find, and they also change with changes in system gain $K$.
- We have no knowledge of the system's performance unless we find the roots of the denominator for specific values of $K$.
- The **Root Locus** will be used to give us a vivid picture of the poles of $T(s)$ as $K$ varies.

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# Defining Root Locus

**Root Locus** is a graphical presentation of the closed-loop poles as **a system parameter** $K$ is varied.

**Root locus** is a powerful method of analysis and design for **transient response** and **stability** of systems of order higher than 2 (but it can also be used for first- and second-order systems) without having to factor the denominator of the closed-loop transfer function.
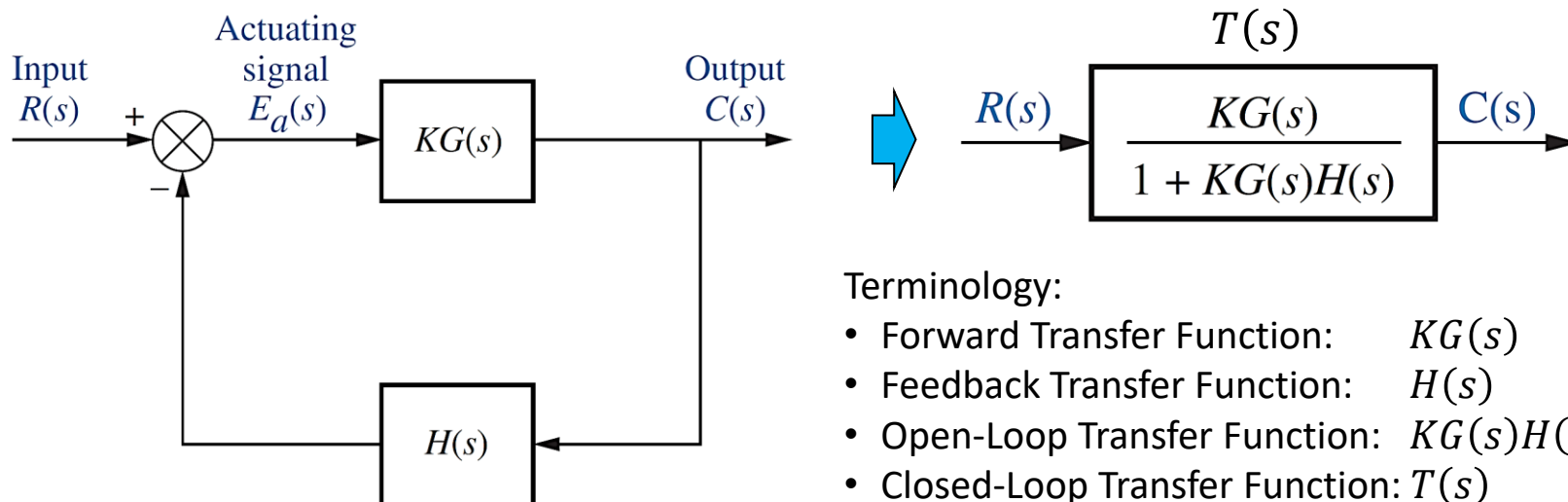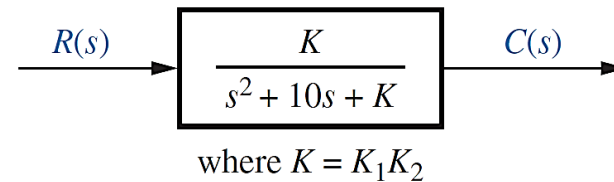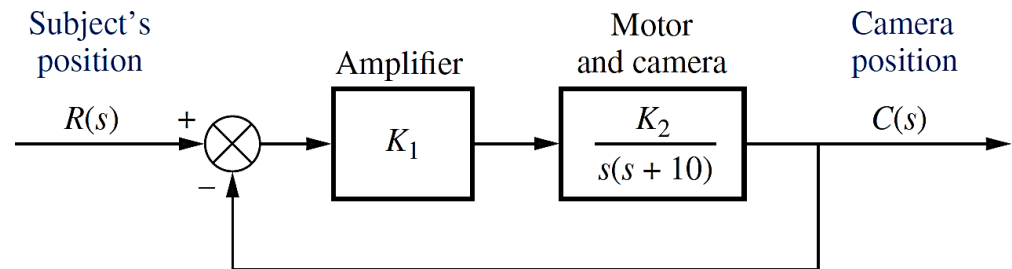


Terminology:
- Forward Transfer Function: $KG(s)$
- Feedback Transfer Function: $H(s)$
- Open-Loop Transfer Function: $KG(s)H(s)$
- Closed-Loop Transfer Function: $T(s)$

# Defining Root Locus: An Example

Consider a camera with motor that can be used to follow moving objects automatically.



| $K$ | Pole 1 | Pole 2 |
|-----|--------|--------|
| 0 | −10 | 0 |
| 5 | −9.47 | −0.53 |
| 10 | −8.87 | −1.13 |
| 15 | −8.16 | −1.84 |
| 20 | −7.24 | −2.76 |
| 25 | −5 | −5 |
| 30 | $-5 + j2.24$ | $-5 - j2.24$ |
| 35 | $-5 + j3.16$ | $-5 - j3.16$ |
| 40 | $-5 + j3.87$ | $-5 - j3.87$ |
| 45 | $-5 + j4.47$ | $-5 - j4.47$ |
| 50 | $-5 + j5$ | $-5 - j5$ |

← Variation of closed-loop poles location for different values of gain $K$.

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# Defining Root Locus: An Example

The closed-loop poles in s-plane:
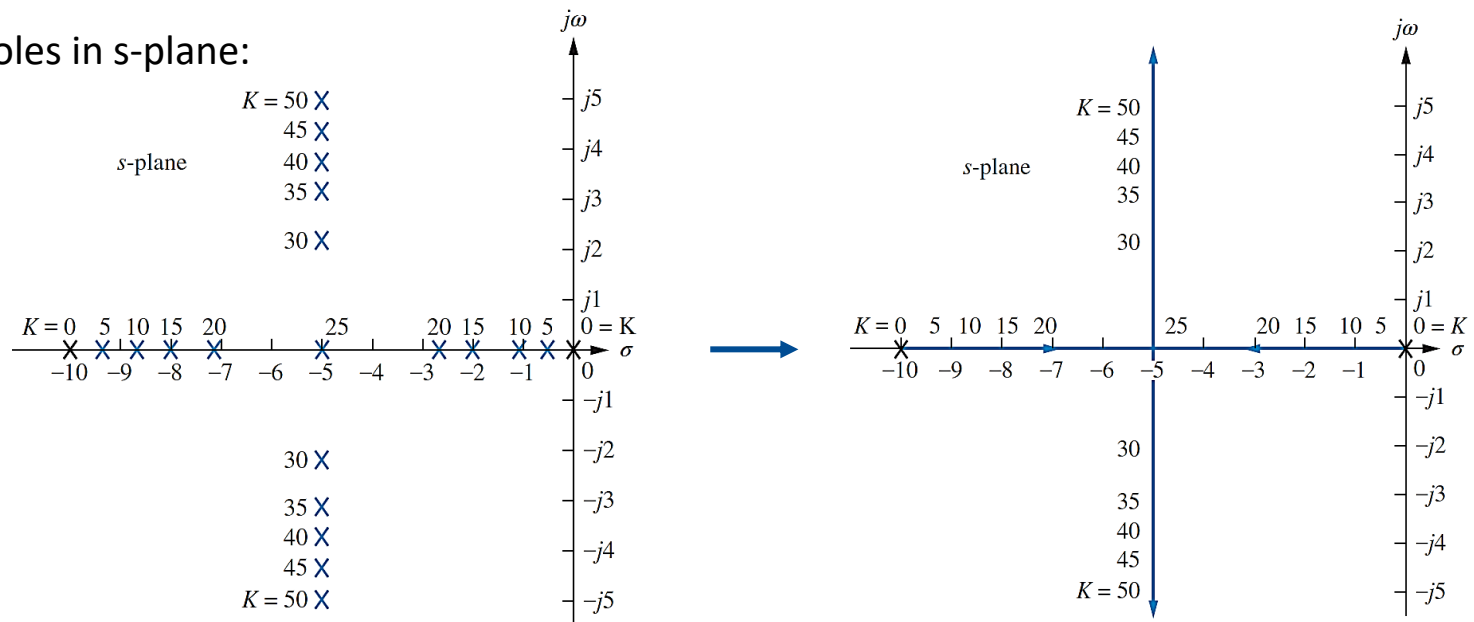


As the gain $K$ increases, the closed-loop pole which is at $-10$ for $K = 0$, moves toward the right, and the closed-loop pole which is at $0$ for $K = 0$, moves toward the left. They meet at $-5$, break away from the real axis, and move into the complex plane. One closed-loop pole moves upward while the other moves downward.

- This representation of the paths (solid lines) of the closed-loop poles as the gain $K$ is varied is called **Root Locus**.

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# Sketching Root Locus

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# Properties of the Root Locus

The closed-loop transfer function for the system is

$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)}$$

For $K \geq 0$, a value of $s$ is a <u>pole</u> of $T(s)$ if

$$KG(s)H(s) = -1 = 1\angle(2k + 1)180° \quad k = 0, \pm1, \pm2, \dots$$
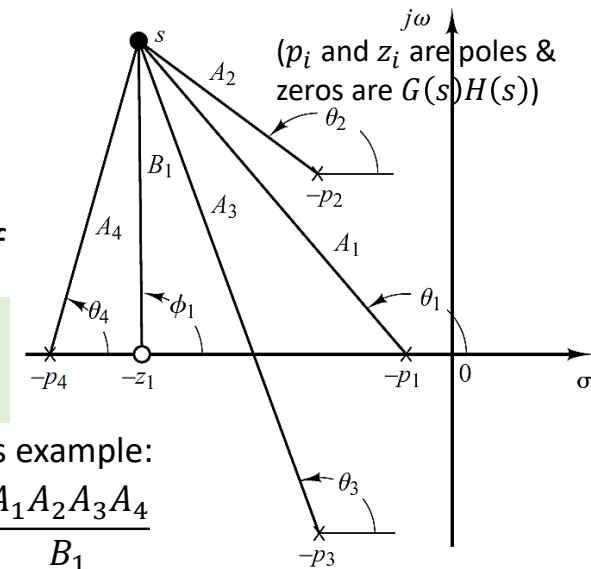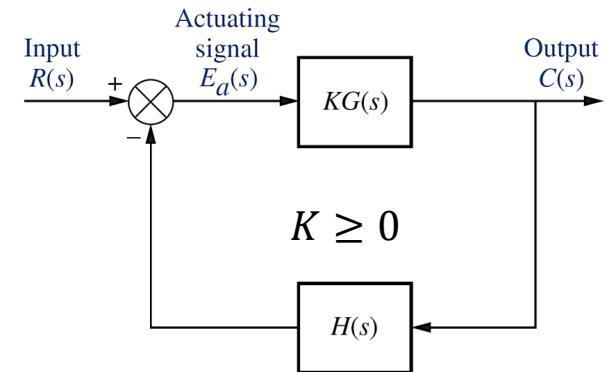
Since $K$ is a positive scalar:

$$\Rightarrow \quad \boxed{\angle G(s)H(s) = (2k + 1)180°} \quad , \quad \boxed{K|G(s)H(s)| = 1}$$

Hence, point $s$ is on the root locus for a particular value of gain $K$ if

$$\angle G(s)H(s) = \sum \begin{array}{c}\text{zero angles of}\\ G(s)H(s)\end{array} - \sum \begin{array}{c}\text{pole angles of}\\ G(s)H(s)\end{array} = (2k + 1)180°$$

$$K = \frac{1}{|G(s)H(s)|} = \frac{\prod \text{pole lengths of } G(s)H(s)}{\prod \text{zero lengths of } G(s)H(s)}$$

(Gain at point $s$)

**Input**
$R(s)$

**Actuating signal**
$E_a(s)$

$KG(s)$

**Output**
$C(s)$

$H(s)$

$$K \geq 0$$

($p_i$ and $z_i$ are poles & zeros are $G(s)H(s)$)

In this example:

$$K = \frac{A_1 A_2 A_3 A_4}{B_1}$$

$$\phi_1 - (\theta_1 + \theta_2 + \theta_3 + \theta_4) = (2k + 1)180°$$

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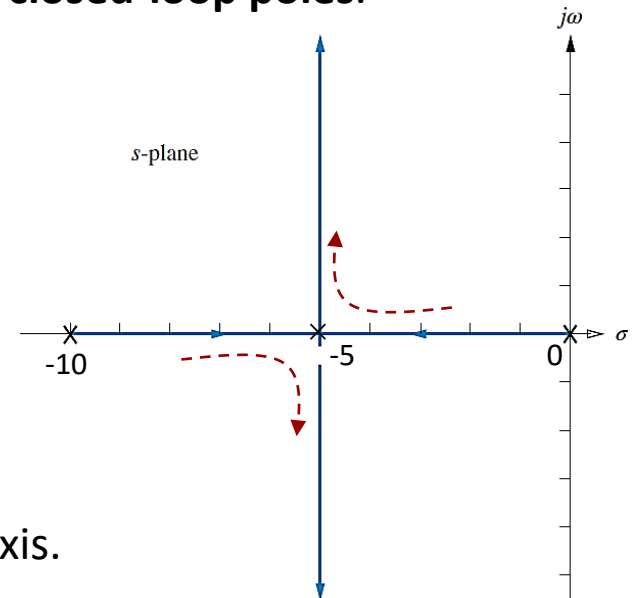
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# Sketching the Root Locus

There are 5 rules that can be used to rapidly **sketch** the root locus with minimal calculations. The sketch gives intuitive insight into the behavior of a control system. Once a sketch is obtained, it is possible to **refine the sketch** and accurately **plot** it by finding actual points or angles on the root locus, which are required for a particular problem.

**1. Number of Branches**: If we define a branch as the path that one pole traverses, the number of branches of the root locus equals the number of **closed-loop poles**.
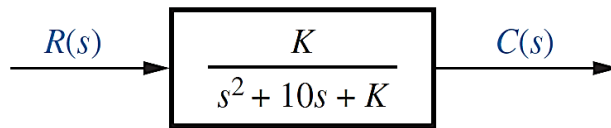


There are **two** branches, one originates at the origin, the other at -10.

**2. Symmetry**: The root locus is symmetrical about the real axis.

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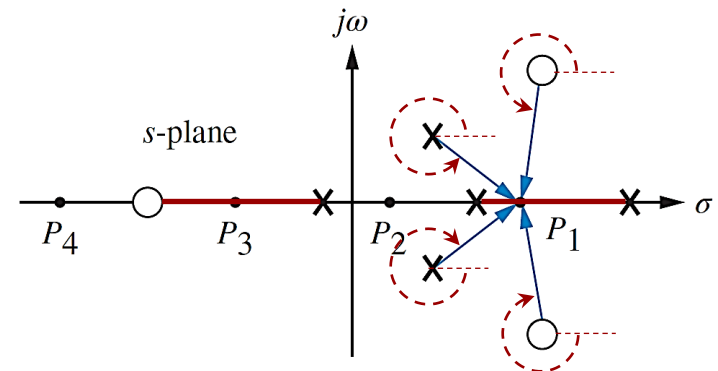Defining Root Locus
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# Sketching the Root Locus

**3. Real-Axis Segments**: Based on the equation

$$\angle G(s)H(s) = \sum_{} \begin{array}{c} \text{zero angles of} \\ G(s)H(s) \end{array} - \sum_{} \begin{array}{c} \text{pole angles of} \\ G(s)H(s) \end{array} = (2k+1)180°$$

at each point $P$ on the real axis $\sigma$:

(1) the angular contribution of a pair of open-loop complex poles or zeros is 0,
(2) the angular contribution of the open-loop poles and zeros to the left of the respective point $P$ is 0,
(3) the angular contribution of the even number of open-loop poles and zeros to the right of the respective point $P$ is 0.

❖ Hence, on the real axis, for $K \geq 0$, the root locus exists to the left of an **odd number** of real-axis, finite open-loop ($G(s)H(s)$) poles and/or zeros.

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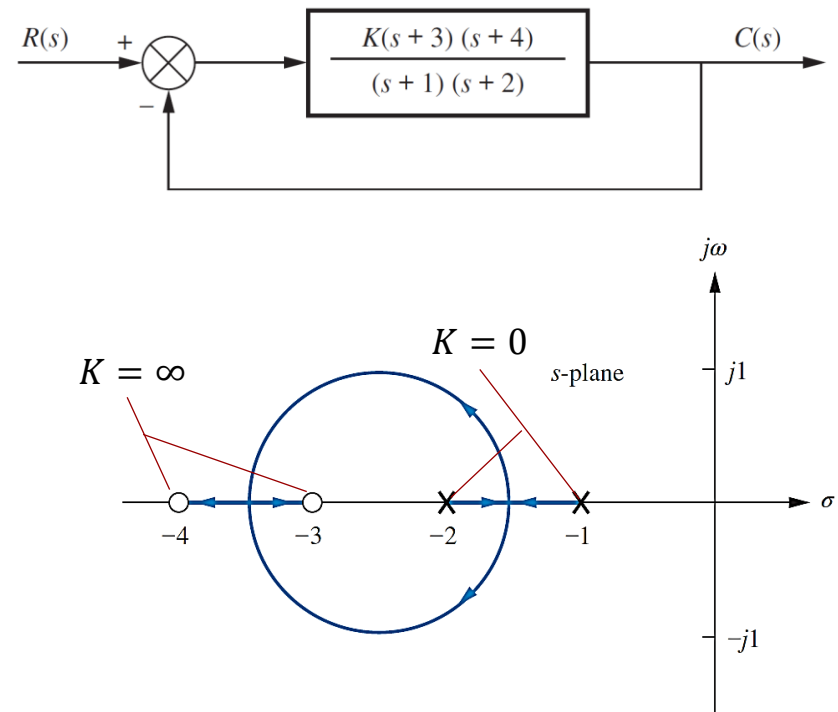Refining Root Locus
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# Sketching the Root Locus

**4. Starting and Ending Points**: The root locus **begins** at the <u>finite and infinite poles</u> of $G(s)H(s)$ where $K = 0$ and **ends** at the <u>finite and infinite zeros</u> of $G(s)H(s)$ where $K = \infty$.
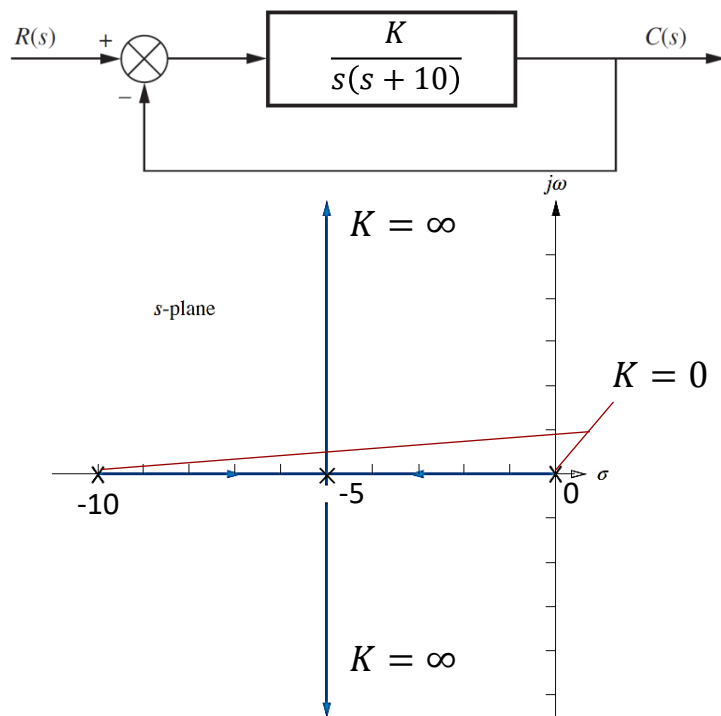


There are two branches that begin at 0 and 10, and end at $\pm\infty$.

There are two branches that begin at -1 and -2, and end at -3 and -4.

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# Sketching the Root Locus

**5. Behavior at Infinity**: This rule tell us what the root locus looks like as it approaches the zeros at infinity or as it moves from the poles at infinity.

- The root locus approaches straight lines as **asymptotes** as the locus approaches infinity. The equation of the asymptotes is given by the real-axis intercept $\sigma_a$ and angle $\theta_a$ as follows:

$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\text{number of finite poles} - \text{number of finite zeros}}$$
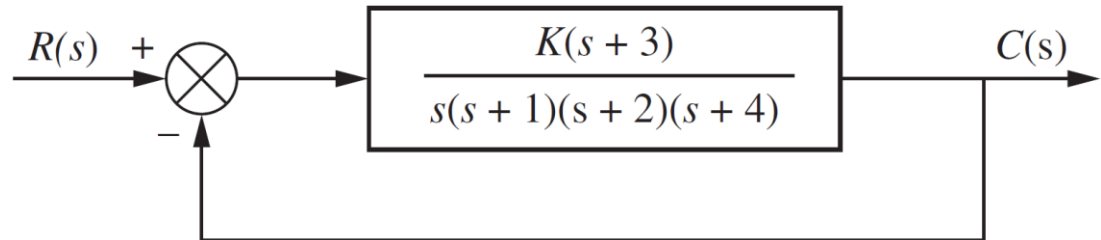
$$\theta_a = \frac{(2k + 1)180°}{\text{number of finite poles} - \text{number of finite zeros}}$$

where the angle $\theta_a$ is given with respect to the positive extension of the real axis and $k = 0, \pm 1, \pm 2, \dots$ yields a multiplicity of lines that account for the many branches of a root locus that approach infinity.

- If the number of poles equals the number of zeros, the root locus does not have any asymptotes.

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# Example

Sketch the root locus for the system.

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# Typical Pole–Zero Configurations and Corresponding Root Loci

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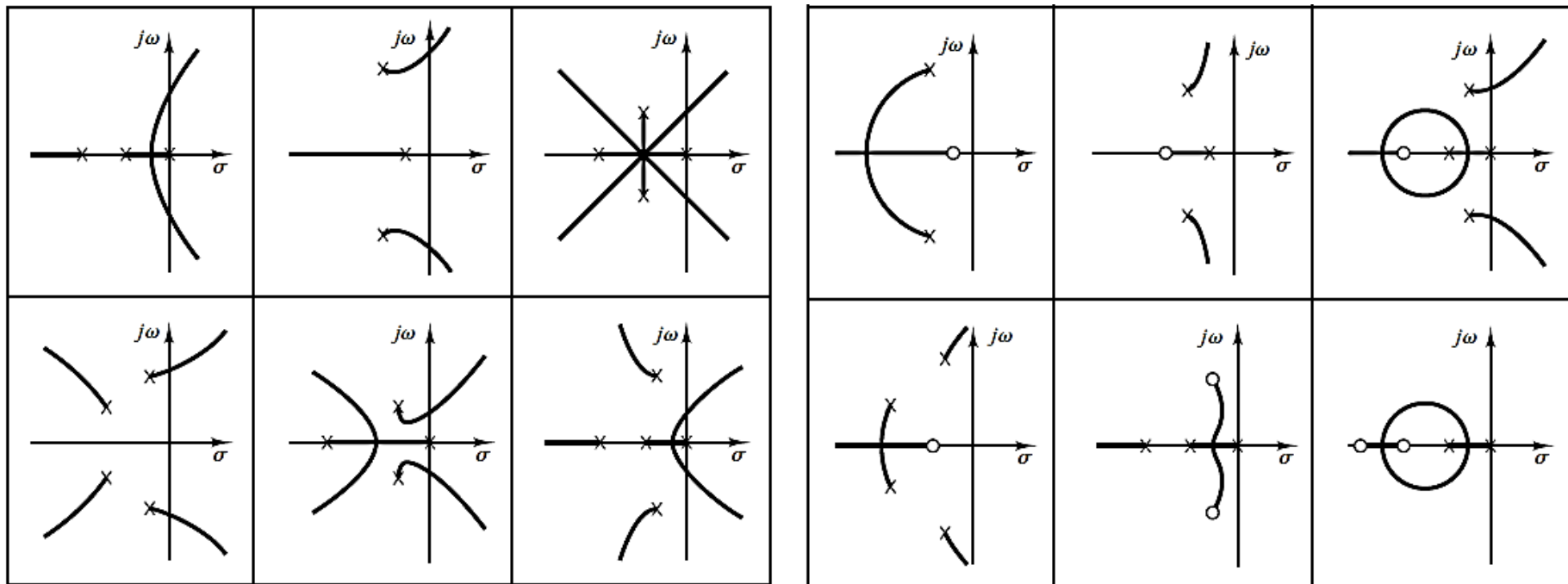
# Refining the Sketch

These 5 rules permit us to **sketch** a root locus rapidly. If we want more detail, we must be able to **accurately find important points** on the root locus along with their associated gain $K$.

**1. Real-Axis Breakaway and Break-In Points**:

The point where the locus leaves the real axis (e.g., $\sigma_1$) is called the **breakaway point**, and the point where the locus returns to the real axis (e.g., $\sigma_2$) is called the **break-in point**.
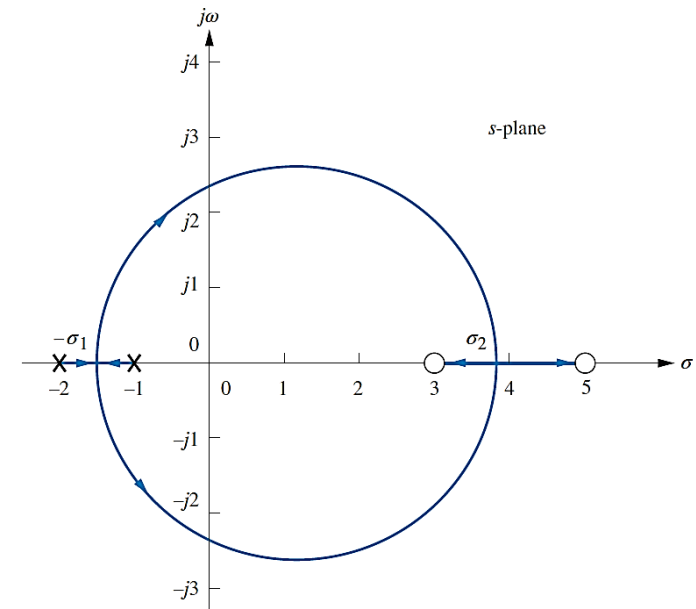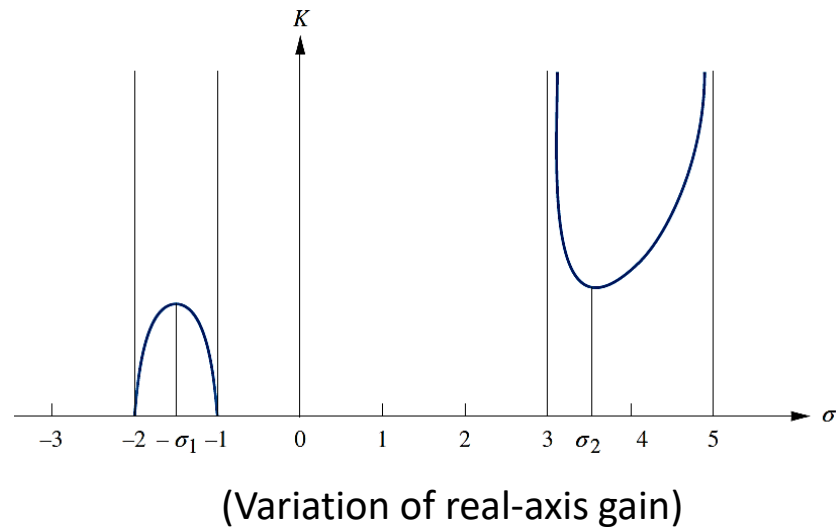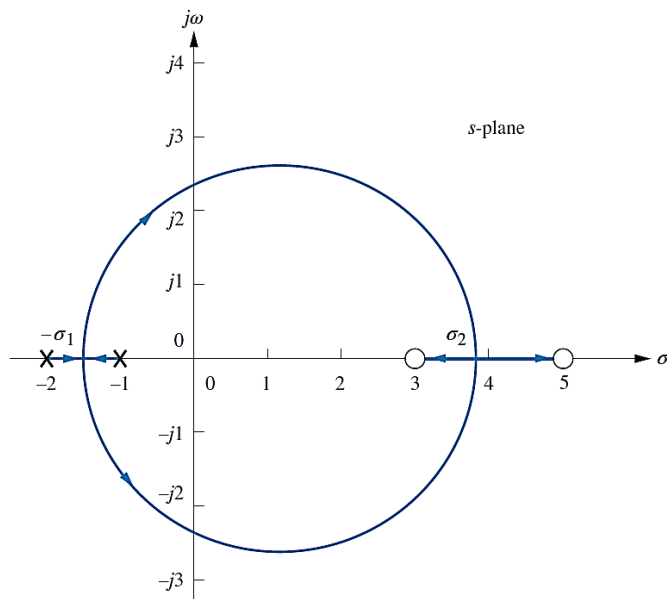
**- Angle with Real Axis**:

At the breakaway or break-in point, the branches of the root locus form an angle of $180°/n$ with the real axis, where $n$ is the number of closed-loop poles arriving at or departing from the **single** breakaway or break-in point on the real axis.

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# Refining the Sketch

**- Location on Real Axis**: The **breakaway** point occurs at a point of **maximum** gain $K$ on the real axis between the open-loop poles. The **break-in** point occurs at a point of **minimum** gain $K$ on the real axis between the open-loop zeros.



(Variation of real-axis gain)

There ate two methods to find the location of these points on the real axis:

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# Refining the Sketch

**Method 1**: For points along the real-axis segment of the root locus where breakaway and break-in points could exist, $s = \sigma$. Hence,

$$K = -\frac{1}{G(\sigma)H(\sigma)}$$

By <u>differentiating</u> this equation with respect to $\sigma$ and setting the derivative equal to zero (i.e., $dK/d\sigma = 0$), the points of maximum and minimum gain $K$, which are the breakaway and break-in points, can be found.
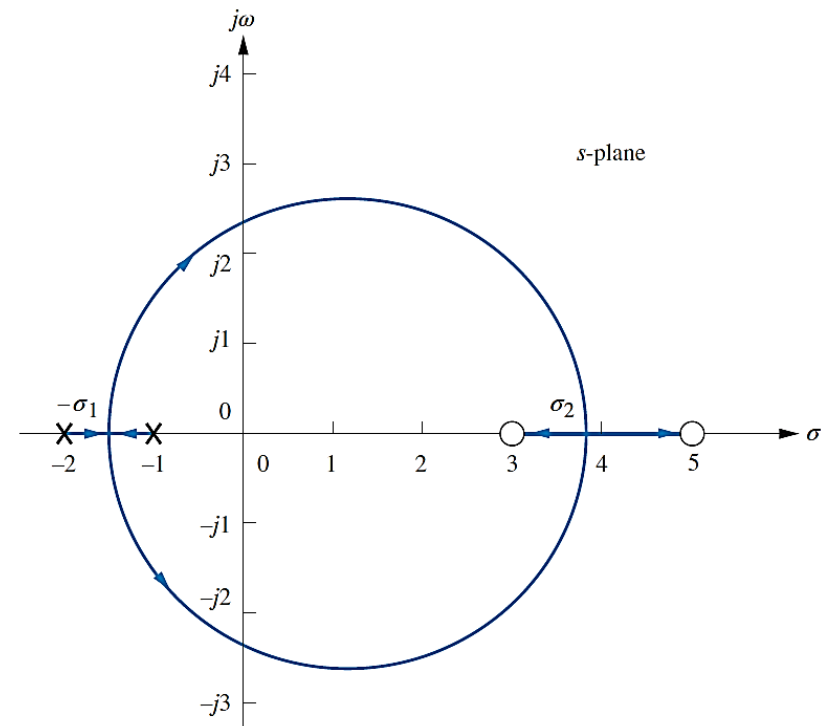
**Method 2**: Breakaway and break-in points satisfy the relationship:

$$\sum_{i=1}^{m} \frac{1}{\sigma + z_i} = \sum_{i=1}^{n} \frac{1}{\sigma + p_i}$$

$m$: is number of zeros
$n$: is number of poles

$$G(s)H(s) = \frac{\prod_{i=1}^{m}(s + z_i)}{\prod_{j=1}^{n}(s + p_j)}$$

By solving this equation for $\sigma$, points of maximum and minimum gain $K$, which are the breakaway and break-in points, can be found <u>without differentiating</u>.

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# Example

Find the breakaway and break-in points for the following root locus.



**Answer**: $\sigma = -1.45$, $\sigma = 3.82$

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# Refining the Sketch

**2. Imaginary-Axis ($j\omega$-Axis) Crossings**:

The $j\omega$-axis crossing is a point on the root locus that separates the stable operation of the system from the unstable operation. The value of $\omega$ at the $j\omega$-axis crossing yields the frequency of oscillation at marginally stable operation.
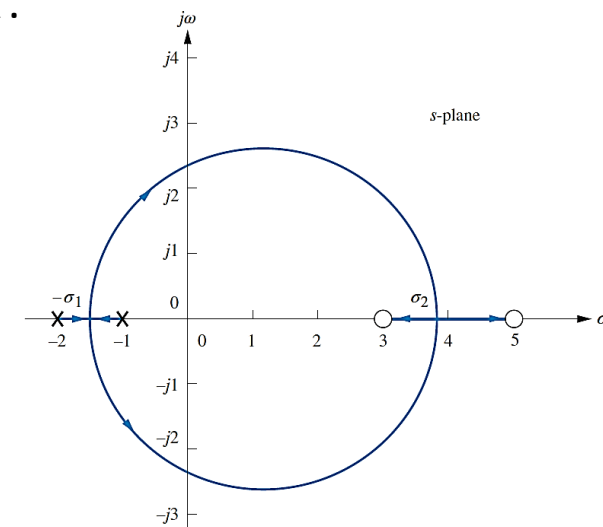There are two methods to find the points where the root loci cross the $j\omega$-axis:

**Method 1**:
Using **Routh-Hurwitz Criterion** for $1 + KG(s)H(s) = 0$ as follows:
- Forcing a row of zeros in the Routh table will yield the gain $K$.
- Solving the even auxiliary polynomial for the roots yields the frequency $\omega$ at the $j\omega$-axis crossing.

$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)}$$

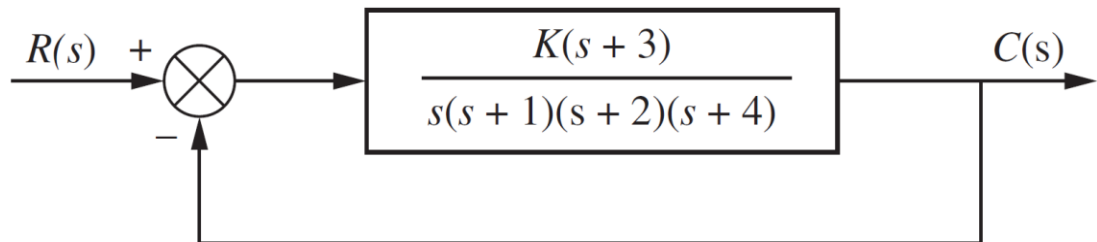**Method 2**: Let $s = j\omega$ in the equation $1 + KG(s)H(s) = 0$, equate both the real part and the imaginary part to zero, and then solve for $\omega$ and $K$.

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# Example

For the following system, find the frequency and gain $K$ for which the root locus crosses the imaginary axis. For what range of $K$ is the system stable?

$$R(s) \xrightarrow{\;+\;} \bigotimes \xrightarrow{\;-\;} \boxed{\dfrac{K(s + 3)}{s(s + 1)(s + 2)(s + 4)}} \longrightarrow C(s)$$

**Answer**: $K = 9.65, \omega = 1.59, 0 \le K < 9.65$

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# Refining the Sketch

**3. Angles of Departure and Arrival** (departure from complex poles and arrival to complex zeros):
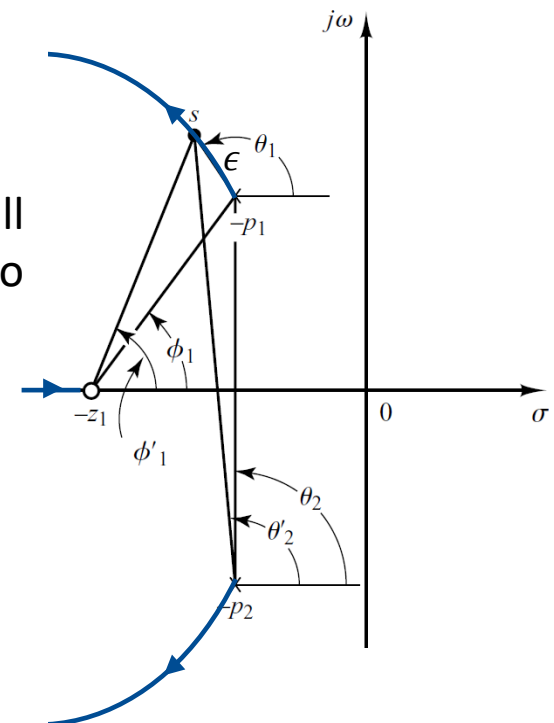
Consider a test point $s$ on the root locus in the very vicinity $\epsilon$ of a complex pole/zero. The sum of angles drawn from all finite poles and zeros to this test point must be an odd multiple of 180°.

In this example:  $\phi'_1 - (\theta_1 + \theta'_2) = 180°(2k + 1)$

Since $\epsilon \to 0$, to find departure/arrival angle, it is assumed that all angles drawn from all other poles and zeros are drawn directly to the pole/zero that is near the test point $s$.
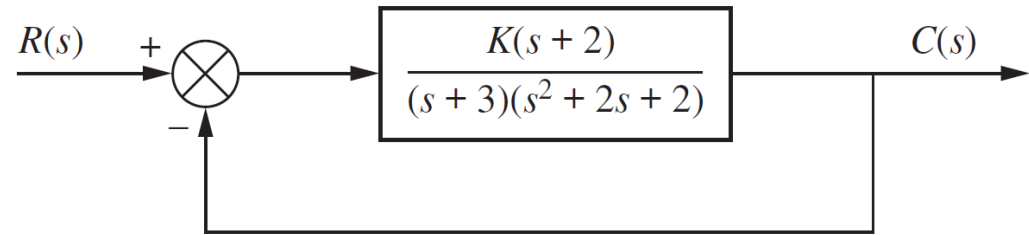
In this example, as $\epsilon \to 0$, $\theta'_2 \to \theta_2$ and $\phi'_1 \to \phi_1$, therefore,

$$\theta_1 = 180°(2k + 1) - \theta_2 + \phi_1$$

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# Example

Given the following unity feedback system, find the angle of departure from the complex poles and sketch the root locus.



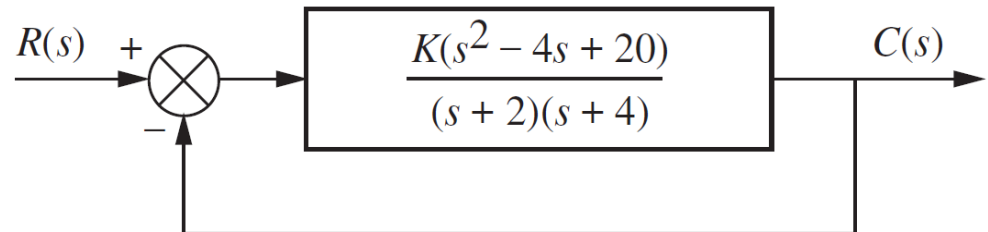$R(s)$  +  $\bigotimes$  $-$  $\dfrac{K(s+2)}{(s+3)(s^2+2s+2)}$  $C(s)$

**Answer**: $\theta = -251.6°$ or $108.4°$

# Example

Sketch the root locus for the system and find the following:

**a**. The exact point and gain where the locus crosses the $j\omega$-axis.

**b**. The range of $K$ within which the system is stable.

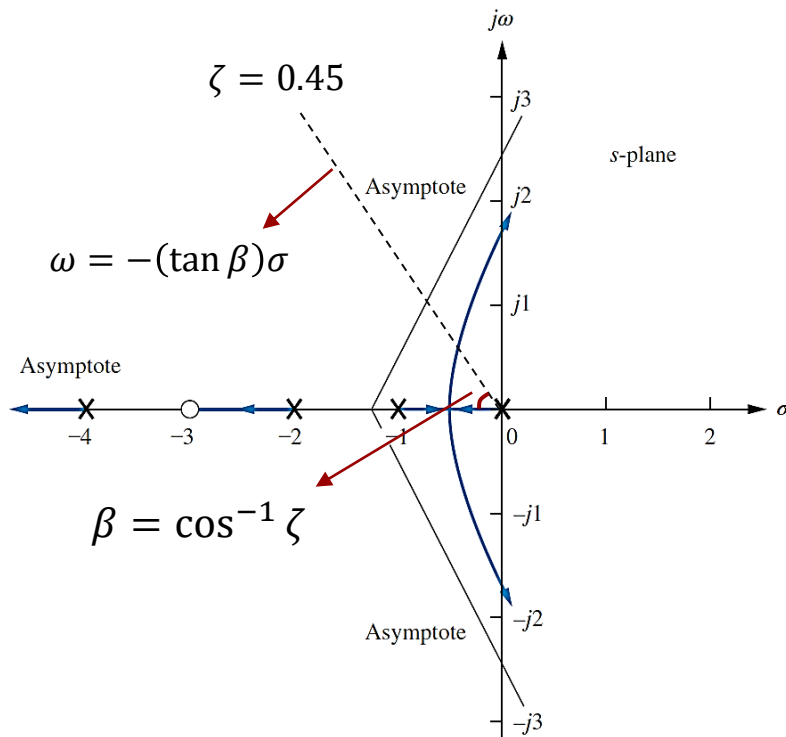**c**. The breakaway point on the real axis.

$$R(s) \xrightarrow{+} \bigotimes \xrightarrow{} \boxed{\frac{K(s^2 - 4s + 20)}{(s + 2)(s + 4)}} \xrightarrow{} C(s)$$

**Answer**: (a) $K = 1.5$, $\omega = 3.9$, (b) $0 \leq K < 1.5$, (c) $\sigma_a = -2.88$

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# Refining the Sketch

**4. Locating a Point on Root Locus and Finding its Associated Gain $K$:**

We may want to accurately locate some points on the root locus as well as find their associated gain $K$. For example, finding a pair of dominant complex-conjugate closed-loop poles such that $\zeta = 0.45$.



$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)}$$

$$1 + KG(s)H(s)\Big|_{s=\sigma+(-\tan\beta)\sigma j} = 0$$

$$\Rightarrow \quad K, \sigma \text{ is calculated.}$$

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# Generalized Root Locus

Sometimes we want to know how the closed-loop poles change as a function of <u>another parameter</u> (not the forward-path gain $K$). For example, in this system the parameter of interest is the open-loop pole at $p_1$.

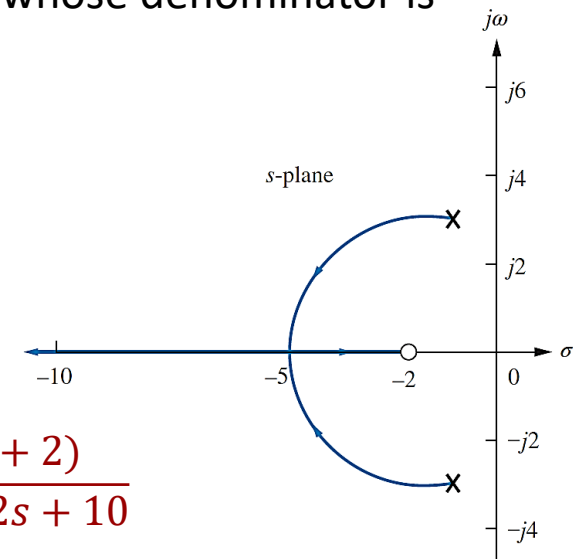$$R(s) \xrightarrow{+} \bigotimes \xrightarrow{} \boxed{\dfrac{10}{(s+2)(s+p_1)}} \xrightarrow{} C(s)$$
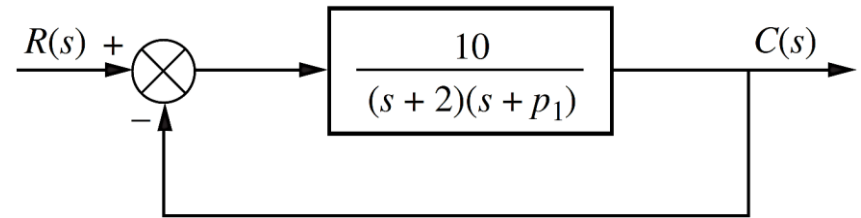
The solution to this problem is to create an equivalent system whose denominator is in the form of $1 + p_1 G(s)H(s)$.

$$T(s) = \frac{10}{s^2 + (p_1 + 2)s + 2p_1 + 10} = \frac{10}{s^2 + 2s + 10 + p_1(s+2)}$$

$$T(s) = \frac{\dfrac{10}{s^2 + 2s + 10}}{1 + p_1 \dfrac{(s+2)}{s^2 + 2s + 10}} \equiv \frac{G(s)}{1 + p_1 G(s)H(s)}$$

$$\Rightarrow \quad G(s)H(s) = \frac{(s+2)}{s^2 + 2s + 10}$$

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# Using MATLAB and Control System Toolbox

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# **Plotting Root Locus Using** rlocus

```
num=[1 2];
den=[1 2 3];
GH = tf(num,den);

rlocus(GH)
% or
% K = 0:0.001:10;
% rlocus(GH,K)

zeta = 0.3:0.3:0.9;
wn = 0:1:4;
sgrid(zeta,wn)
```

$$G(s)H(s) = \frac{s+2}{s^2 + 2s + 3}$$

The root locus can be drawn over a grid that shows constant damping ratio ($\zeta$) and constant natural frequency ($\omega_n$) curves.