Ch2: An Introductory Example

2R Planar Manipulator

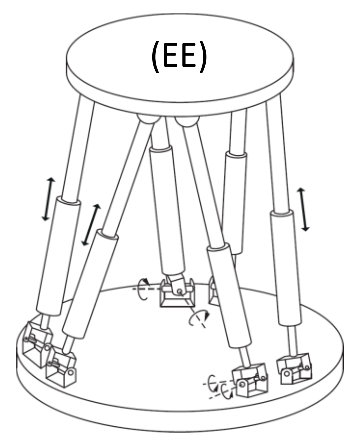
Robot Mechanical Structure

- A robot is mechanically constructed by connecting a set of bodies, called **Links**, to each other using various types of **Joints**.

  * All the robots considered in this course have links that can be modeled as **rigid bodies**.

- **Actuators**, such as electric motors, deliver forces and torques to the joints, thereby causing motion of the robot.

- An **End-Effector** (EE), such as a gripper or hand for grasping and manipulating objects, is attached to a specific link.



2R (or RR) Planar Manipulator

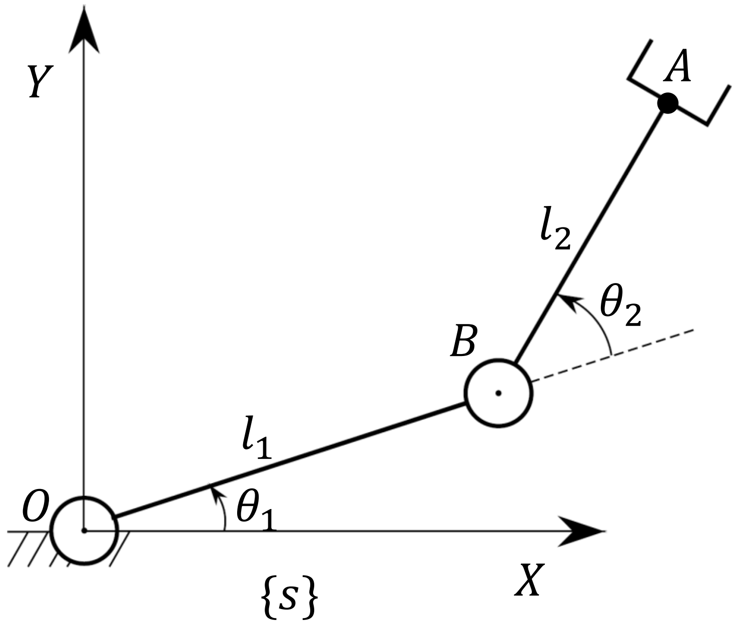
(θ_1, θ_2): Joint angles (or joint positions)

(x, y): Position of end-effector (point A)

$\{s\}$: Base frame of manipulator

l_1: Length of link 1

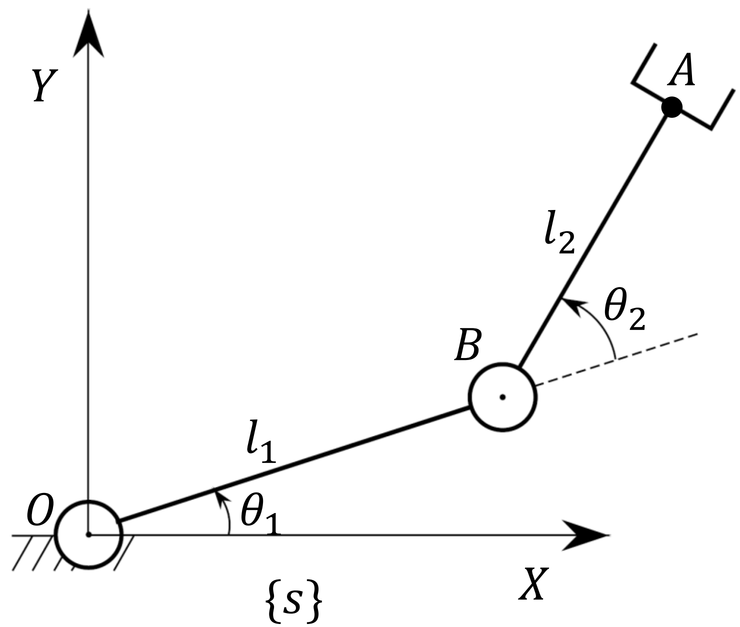
l_2: Length of link 2



Position Kinematics

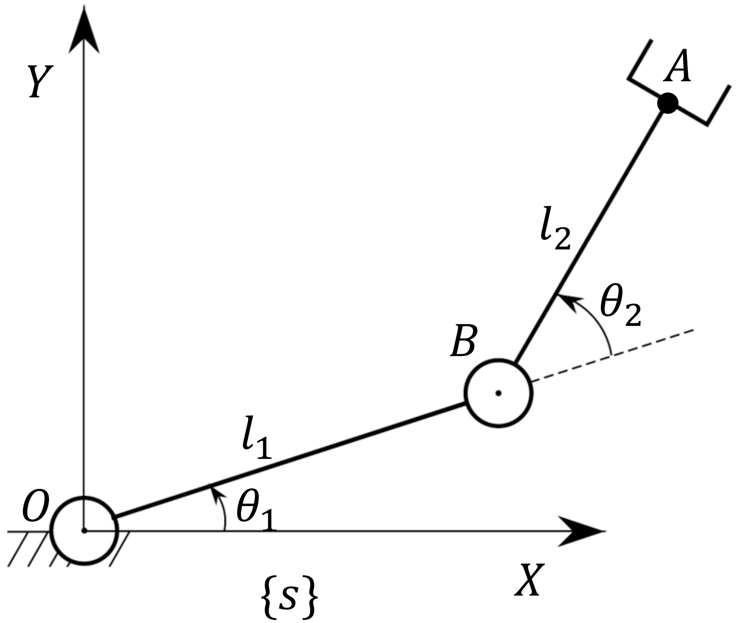
Relation between Joint Angles and End-Effector Position

What is the relationship between the joint angles, (θ_1, θ_2), and the position of the end effector point A, (x, y), in the base frame $\{s\}$?



Forward (Direct) Position Kinematics

Given the joint angles, (θ_1, θ_2), of the 2R robot, find the position, (x, y) of the end-effector point A, in the base frame $\{s\}$.



Forward (Direct) Position Kinematics

$\boldsymbol{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$: Vector of joint angles

$\boldsymbol{q} = \begin{bmatrix} x \\ y \end{bmatrix}$: Position vector of end-effector point

$$x = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) \equiv f_1(\theta_1, \theta_2)$$
$$y = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) \equiv f_2(\theta_1, \theta_2)$$

$$\boldsymbol{q} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f_1(\theta_1, \theta_2) \\ f_2(\theta_1, \theta_2) \end{bmatrix}$$

More abstractly, the forward kinematics map is

$$\boldsymbol{q} = \boldsymbol{f}(\boldsymbol{\theta})$$

$$\boldsymbol{f}(\boldsymbol{\theta}) = \begin{bmatrix} f_1(\theta_1, \theta_2) \\ f_2(\theta_1, \theta_2) \end{bmatrix}$$

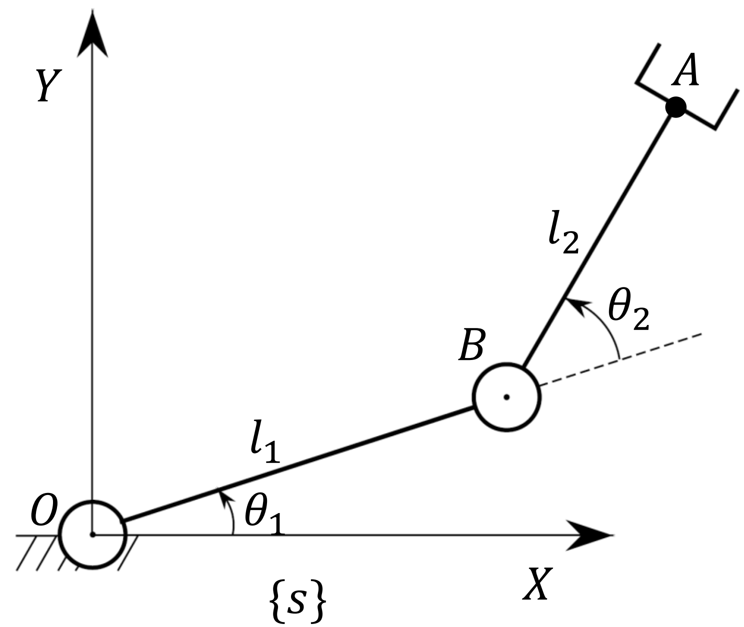
where $\boldsymbol{f}$ is a vector function.

Inverse Position Kinematics

Given the position, (x, y), of the end effector point A, find the joint angles, (θ_1, θ_2) so that the position (x, y) is reached.

In other words, from the equations

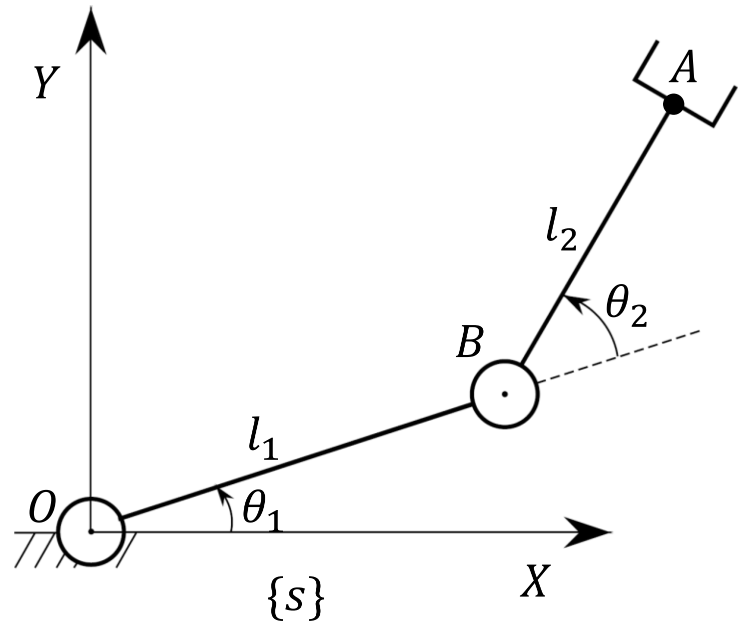
$$x = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2)$$
$$y = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2)$$



Find θ_1 and θ_2 as a function of x and y.

Numerical Example (Exercise)

Forward and Inverse Position Kinematics:



$$x = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2)$$
$$y = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2)$$

$$\theta_2 = \operatorname{atan2}\left(\pm\sqrt{1-u^2}, u\right)$$
$$\theta_1 = \operatorname{atan2}(y, x) - \operatorname{atan2}(l_2 \sin\theta_2, l_1 + l_2 \cos\theta_2)$$

$$u = \frac{x^2 + y^2 - l_1^2 - l_2^2}{2 l_1 l_2}$$

Velocity Kinematics

Relation between Joint Angle Rates and End-Effector Velocity

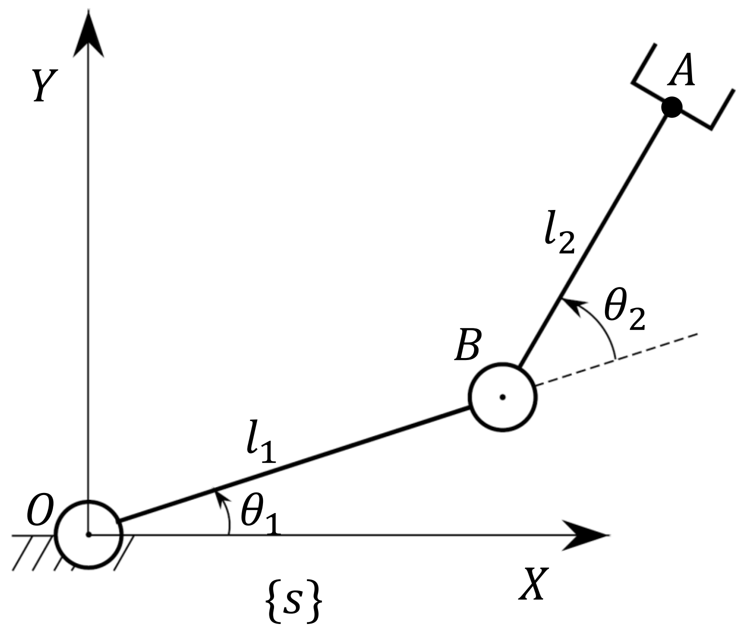
What is the relationship between the joint angle rates of motion (or joint velocities) $(\dot{\theta}_1, \dot{\theta}_2)$, and the velocity of the end effector point (v_x, v_y)?

$\dot{\theta}_1 = \dfrac{d\theta_1}{dt}$: Rate of change of angle of joint 1.

$\dot{\theta}_2 = \dfrac{d\theta_2}{dt}$: Rate of change of angle of joint 2.

$v_x = \dfrac{dx}{dt} = \dot{x}$: x-component of velocity of point A.

$v_y = \dfrac{dy}{dt} = \dot{y}$: y-component of velocity of point A.



Relation between Joint Angle Rates and End-Effector Velocity

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1+\theta_2) & -l_2 \sin(\theta_1+\theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) & l_2 \cos(\theta_1+\theta_2) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

The manipulator (analytic) Jacobian is:

$$\boldsymbol{J}(\theta_1,\theta_2) = \begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1+\theta_2) & -l_2 \sin(\theta_1+\theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) & l_2 \cos(\theta_1+\theta_2) \end{bmatrix}$$

$\dot{\boldsymbol{\theta}} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$: Vector of joint angle rates.

$\boldsymbol{v} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$: Velocity of end-effector point.

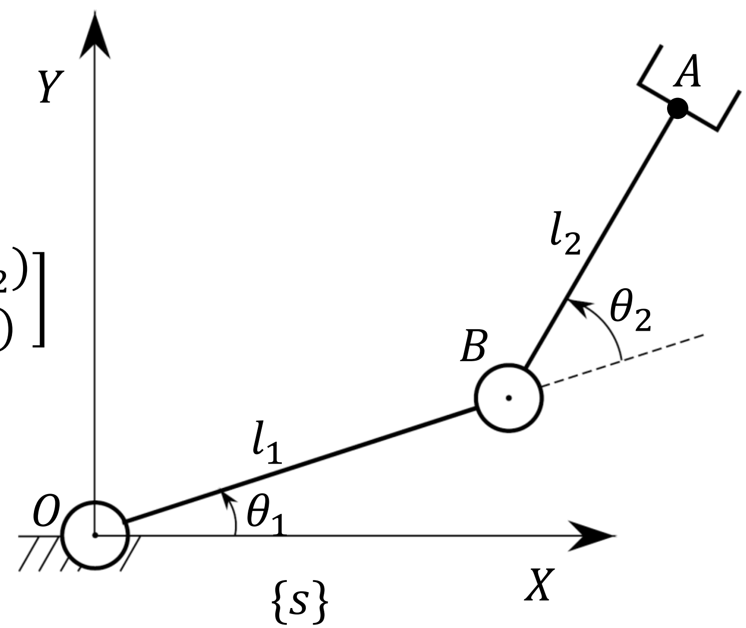
The velocity kinematics equations in vector-matrix form is: $\boldsymbol{v} = \boldsymbol{J}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$

Forward (Direct) Velocity Kinematics

Given the configuration of the robot, $\boldsymbol{\theta}$, and the joint angle rates, $\dot{\boldsymbol{\theta}}$, compute the velocity, $\boldsymbol{v}$ of the end effector.

$$\boldsymbol{v} = \boldsymbol{J}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$$

$$\boldsymbol{J}(\boldsymbol{\theta}) = \begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1+\theta_2) & -l_2 \sin(\theta_1+\theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) & l_2 \cos(\theta_1+\theta_2) \end{bmatrix}$$

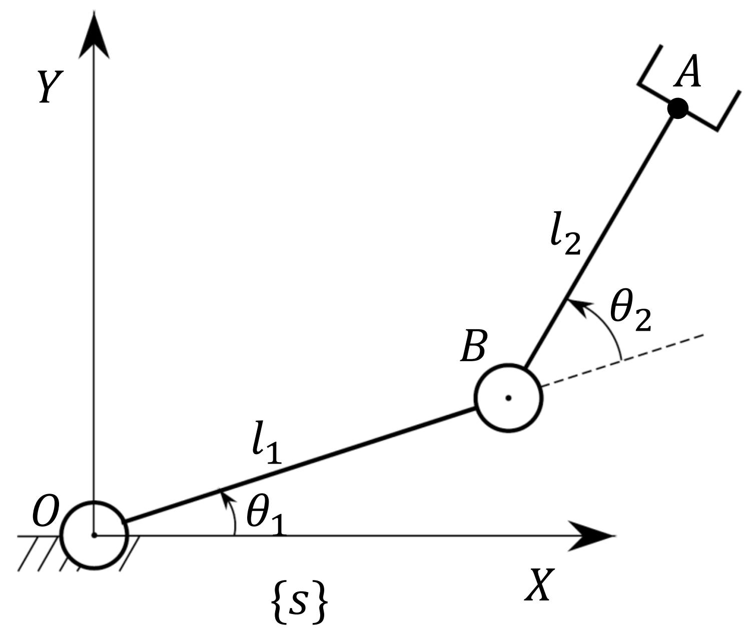


Inverse Velocity Kinematics

Given the configuration of the robot, $\boldsymbol{\theta}$, and the velocity, $\boldsymbol{v}$, of the end effector, compute the joint angle rates, $\dot{\boldsymbol{\theta}}$.

$$\dot{\boldsymbol{\theta}} = \boldsymbol{J}^{-1}(\boldsymbol{\theta})\boldsymbol{v}$$

assuming $\boldsymbol{J}^{-1}(\boldsymbol{\theta})$ exists or the Jacobian matrix is invertible at the configuration $\boldsymbol{\theta}$.

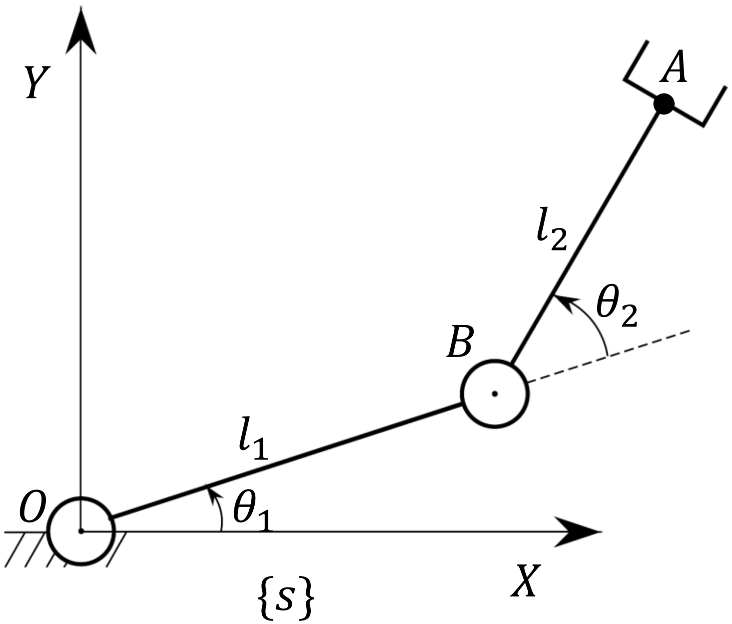


Kinematic Singularities

The configuration $\boldsymbol{\theta}$ at which the Jacobian, $\boldsymbol{J}(\boldsymbol{\theta})$ of a manipulator loses rank is called a **kinematic singularity** or **singular configuration** of the manipulator.

For a 2R manipulator, the Jacobian, $\boldsymbol{J}(\boldsymbol{\theta})$ losing rank implies $\det\big(\boldsymbol{J}(\boldsymbol{\theta})\big) = 0$.

$$\boldsymbol{J}(\boldsymbol{\theta}) = \begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{bmatrix}$$

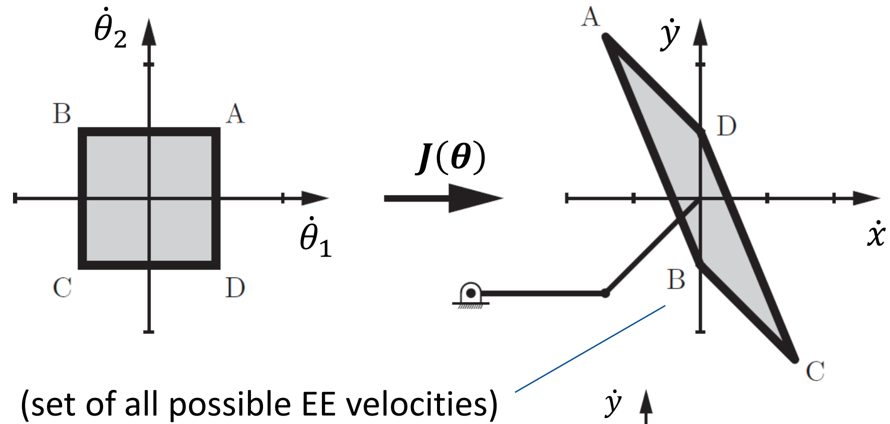


Physical Implications of Kinematic Singularities

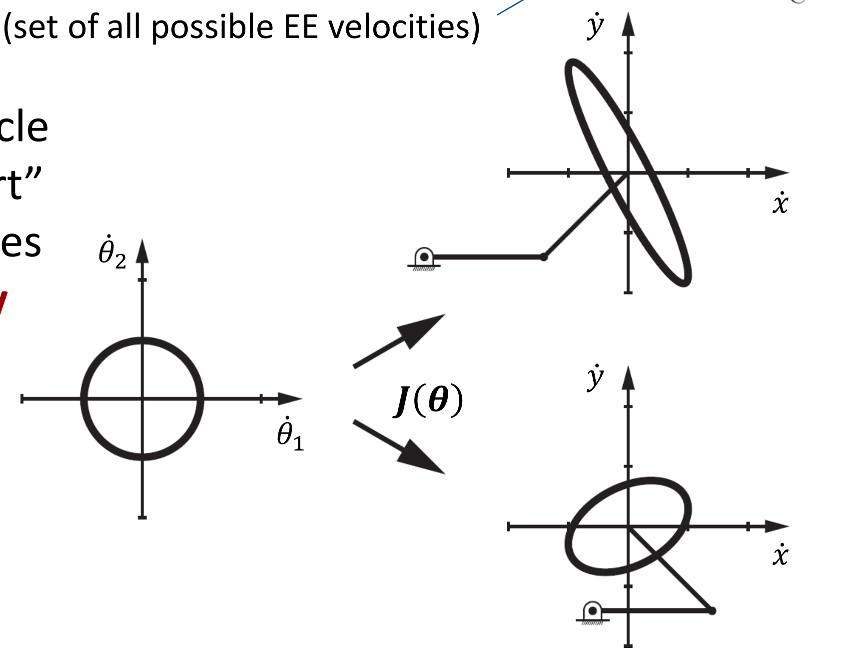
Why should we care about singular (or almost singular) configurations?

Velocity Manipulability Ellipsoid

The Jacobian can be used to map bounds on the rotational speed of the joints (which is a polygon) to bounds on $\boldsymbol{v}$.



The Jacobian can be also used to map a unit circle of joint velocities in the $\theta_1 - \theta_2$-plane ("iso-effort" contour) to an ellipse in the space of EE velocities (this ellipse is called the **velocity manipulability ellipsoid/ellipse**).



The closer the ellipsoid is to a circle, the more easily can the tip move in arbitrary directions.

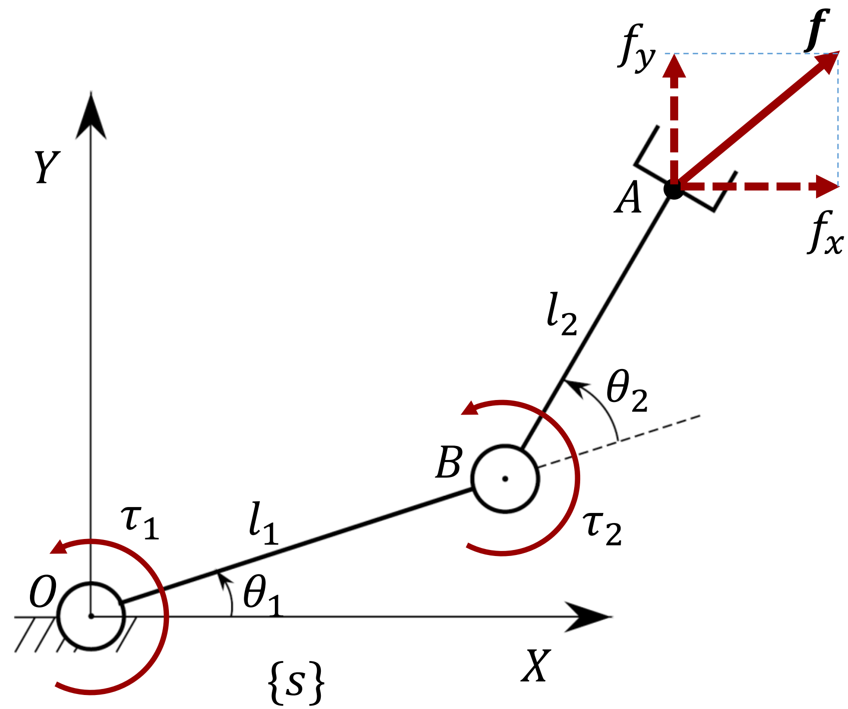
Statics

Statics

What is the relationship between the applied force $\boldsymbol{f}$ and the joint torques $\boldsymbol{\tau}$ such that the manipulator is at equilibrium at a given configuration $\boldsymbol{\theta}$?

$\boldsymbol{f} = \begin{bmatrix} f_x \\ f_y \end{bmatrix}$: Force acting at end-effector point A

$\boldsymbol{\tau} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$: Vector of joint torques required to resist $\boldsymbol{f}$



(Assume that gravitational acceleration $\boldsymbol{g}$ is $\mathbf{0}$ or the robot is horizontal)

Statics

$$\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = \begin{bmatrix}-l_1\sin\theta_1 - l_2\sin(\theta_1+\theta_2) & l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2)\\ -l_2\sin(\theta_1+\theta_2) & l_2\cos(\theta_1+\theta_2)\end{bmatrix}\begin{bmatrix}f_x\\ f_y\end{bmatrix}$$

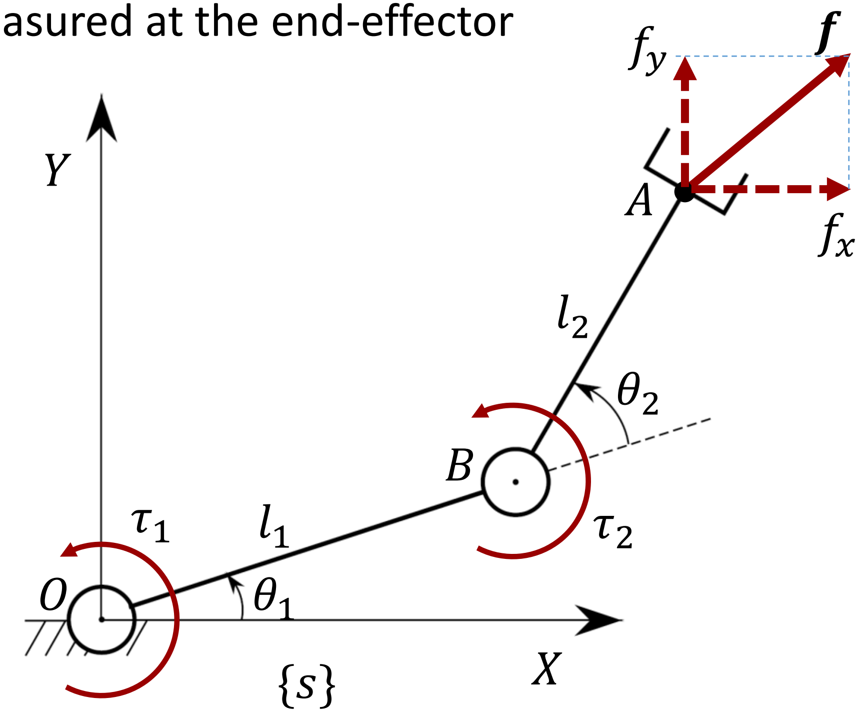
$$\boldsymbol{\tau} = \boldsymbol{J}(\boldsymbol{\theta})^T\boldsymbol{f}$$

Statics

A more general method to derive a relation between $\boldsymbol{f}$ and $\boldsymbol{\tau}$.

Principle of conservation of power:

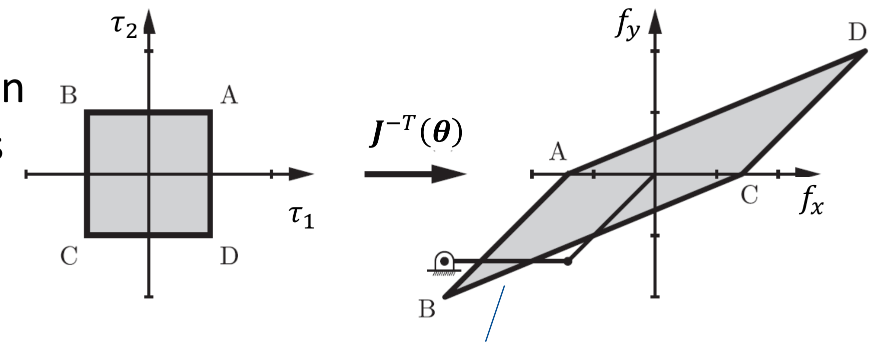
power generated at the joints = power measured at the end-effector



(Assume that $\boldsymbol{g} = \mathbf{0}$)

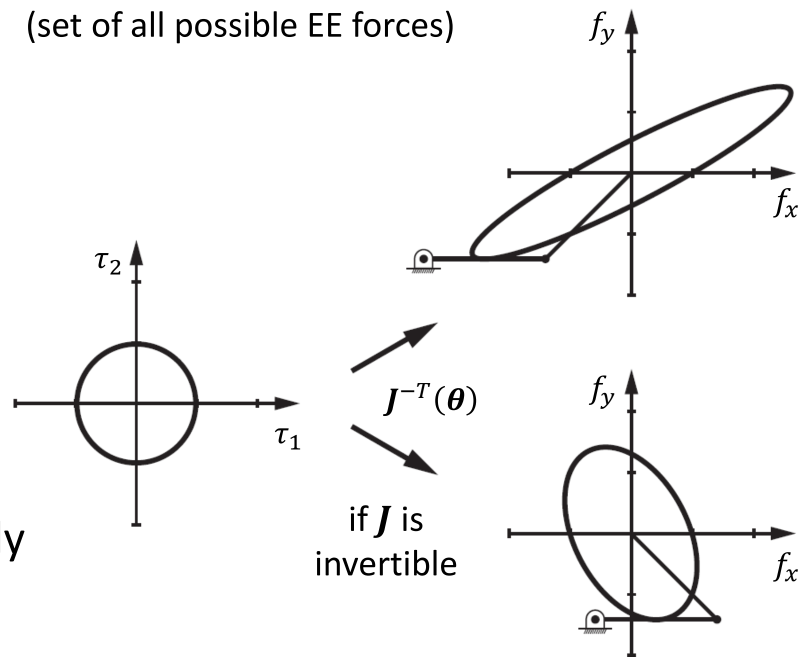
Force Manipulability Ellipsoid

Since $\boldsymbol{f} = (\boldsymbol{J}(\boldsymbol{\theta})^T)^{-1}\boldsymbol{\tau} = \boldsymbol{J}(\boldsymbol{\theta})^{-T}\boldsymbol{\tau}$, Jacobian transpose inverse can be used to map bounds on the joint torques (which is a polygon) to bounds on end-effector force $\boldsymbol{f}$.



(set of all possible EE forces)

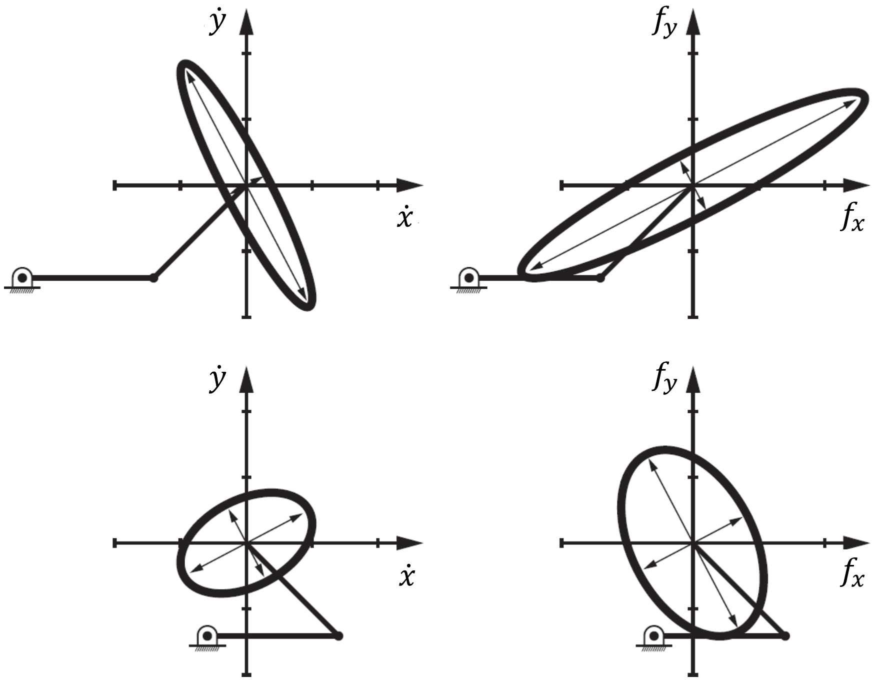
The Jacobian transpose inverse can be also used to map a unit circle of joint torques in the $\tau_1 - \tau_2$-plane ("iso-effort" contour) to an ellipse in the space of EE forces (this ellipse is called the **force manipulability ellipsoid/ellipse**).



The closer the ellipsoid is to a circle, the more easily can the EE generate forces in arbitrary directions.

Kineto-Statics Duality

If it is easy to generate a tip velocity in a given direction then it is difficult to generate a force in that same direction, and vice versa.



$$\boldsymbol{v} = \boldsymbol{J}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$$

$$\boldsymbol{\tau} = \boldsymbol{J}(\boldsymbol{\theta})^T \boldsymbol{f}$$

At a singularity, EE motion capability becomes zero in one or more directions, and it can resist infinite force in one or more directions.