Ch4: Rigid-Body Motion – Rotation

Reference Frames
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Rotation Matrices
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Angular Velocity
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Exponential Coordinate Representation
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Euler Angles
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Unit Quaternion
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Reference Frames

Reference Frames
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Rotation Matrices
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Angular Velocity
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Exponential Coordinate Representation
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Euler Angles
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Unit Quaternion
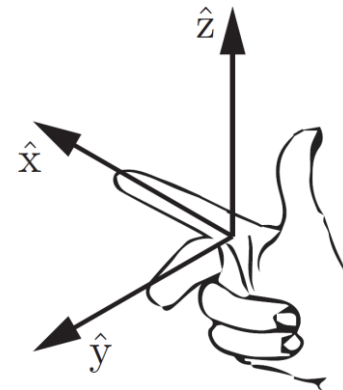○○○○

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Reference Frames

- **Fixed Space Frame** $\{s\}$: A <u>stationary</u>, inertial frame and there is only one.

- **Body-attached Frame**: A frame fixed to a body and moves with it.

- **Body Frame** $\{b\}$: A <u>stationary</u>, inertial frame that is instantaneously coincident with the body-attached frame.
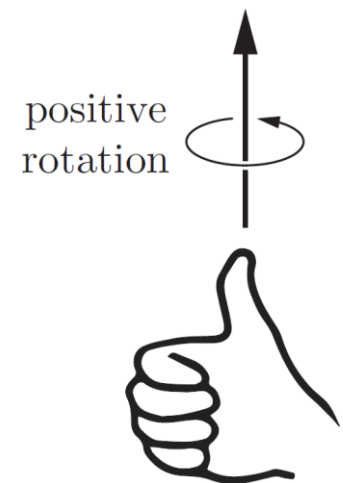
In this course, all frames are instantaneously stationary.

Reference Frames
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Rotation Matrices
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Angular Velocity
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Exponential Coordinate Representation
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Euler Angles
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Unit Quaternion
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Reference Frames

All reference frames are **right-handed**.

A **positive rotation** about an axis is defined as the direction in which the fingers of the right hand curl when the thumb is pointed along the axis.

Reference Frames · · | Rotation Matrices ○○○○○○○○○○○○○○ | Angular Velocity ○○○○ | Exponential Coordinate Representation ○○○○○○○○○○ | Euler Angles ○○○○○ | Unit Quaternion ○○○○

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Rotation Matrices

Rotation in 2D Space

In 2D, the simplest way to describe the orientation of the body frame $\{b\}$ relative to the fixed frame $\{s\}$ is by specifying the angle θ.
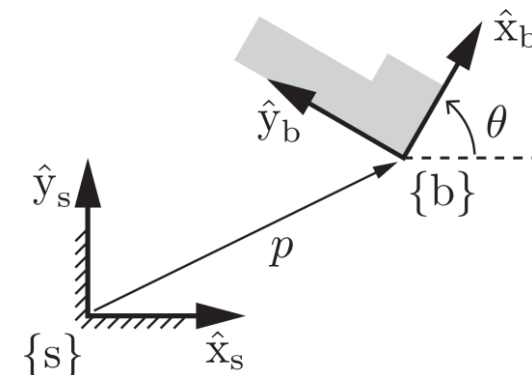
Another way is to specify the directions of the unit axes $\hat{x}_b$ and $\hat{y}_b$ of $\{b\}$ relative to $\{s\}$.

$$\Rightarrow \quad \boldsymbol{R} = [\hat{x}_b \quad \hat{y}_b] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \theta \in [0,2\pi)$$

Rotation Matrix

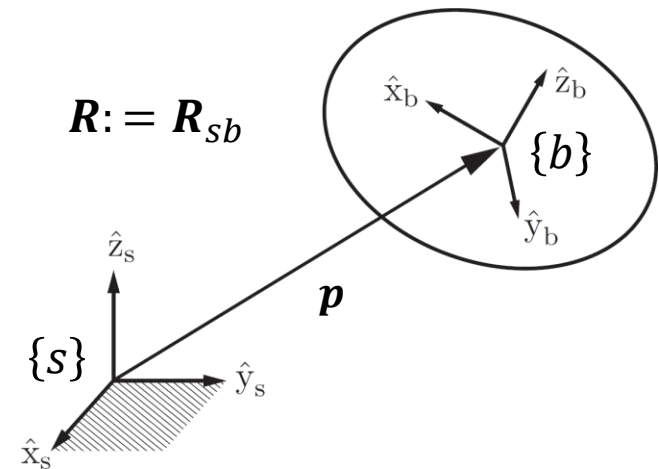$$\boldsymbol{R} := \boldsymbol{R}_{sb}$$

Rotation in 3D Space

In 3D, a way to describe the orientation of the body frame $\{b\}$ relative to the fixed frame $\{s\}$ is by specifying the directions of the unit axes $\hat{x}_b$, $\hat{y}_b$, and $\hat{z}_b$ of $\{b\}$ relative to $\{s\}$.

$$\boldsymbol{R} = [\hat{x}_b \quad \hat{y}_b \quad \hat{z}_b] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \in \mathbb{R}^{3\times3}$$

Rotation Matrix

$$\boldsymbol{R} := \boldsymbol{R}_{sb}$$

This is as Implicit representation the C-space.

Reference Frames   **Rotation Matrices**   Angular Velocity   Exponential Coordinate Representation   Euler Angles   Unit Quaternion

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Constraints on Rotation Matrix

1- The unit norm condition: $\hat{x}_b$, $\hat{y}_b$, and $\hat{z}_b$ are all unit vectors.

2- The orthogonality condition: $\hat{x}_b . \hat{y}_b = \hat{x}_b . \hat{z}_b = \hat{y}_b . \hat{z}_b = 0$

Compact form: $\boldsymbol{R}^T \boldsymbol{R} = \boldsymbol{I}_3$

For right-handed frames: $\det(\boldsymbol{R}) = 1$

Definition of a Group

A **group** is a set of elements $G = \{a, b, c, \dots\}$ and a binary operation • on any two elements satisfying

- **Closure**: $a • b \in G \quad \forall a, b \in G$
- **Associativity**: $(a • b) • c = a • (b • c) \qquad \forall a, b, c \in G$
- **Identity Element Existence**: $\exists I \in G$ such that $a • I = I • a = a \qquad \forall a \in G$
- **Inverse Element Existence**: $\forall a \in G, \exists a^{-1} \in G$ such that $a • a^{-1} = a^{-1} • a = I$

Reference Frames   **Rotation Matrices**   Angular Velocity   Exponential Coordinate Representation   Euler Angles   Unit Quaternion

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Special Orthogonal Group $SO(n)$

The **special orthogonal group** $SO(n)$, $n = 2,3$, also known as the (Lie) group of rotation matrices, is the set of all $n \times n$ real matrices $\boldsymbol{R}$ that satisfy **(i)** $\boldsymbol{R}^T \boldsymbol{R} = \boldsymbol{I}_3$ and **(ii)** $\det(\boldsymbol{R}) = 1$.

orthogonal

special

$SO(2)$ is a subgroup of $SO(3)$: $SO(2) \subset SO(3)$

$\boldsymbol{R} \in SO(3)$ $SO(3) = \{\boldsymbol{R} \in \mathbb{R}^{3 \times 3} | \boldsymbol{R}^T \boldsymbol{R} = \boldsymbol{R} \boldsymbol{R}^T = \boldsymbol{I}_3, \det(\boldsymbol{R}) = 1\}$

Reference Frames
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Rotation Matrices
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Angular Velocity
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Exponential Coordinate Representation
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Euler Angles
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Unit Quaternion
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Properties of Rotation Matrices

$SO(3)$ (or $SO(2)$) is a **matrix (Lie) group** (and the group operation • is matrix multiplication).

- **Closure**: $\quad\quad\quad \boldsymbol{R}_1\boldsymbol{R}_2 \in SO(3)$
- **Associative**: $\quad (\boldsymbol{R}_1\boldsymbol{R}_2)\boldsymbol{R}_3 = \boldsymbol{R}_1(\boldsymbol{R}_2\boldsymbol{R}_3)$ (but generally not commutative, $\boldsymbol{R}_1\boldsymbol{R}_2 \neq \boldsymbol{R}_2\boldsymbol{R}_1$)
- **Identity**: $\quad\quad\quad \exists \boldsymbol{I}_3 \in SO(3)$ such that $\boldsymbol{R}\boldsymbol{I}_3 = \boldsymbol{I}_3\boldsymbol{R} = \boldsymbol{R}$
- **Inverse**: $\quad\quad\quad \exists\, \boldsymbol{R}^{-1} \in SO(3)$ such that $\boldsymbol{R}\boldsymbol{R}^{-1} = \boldsymbol{R}^{-1}\boldsymbol{R} = \boldsymbol{I}_3 \quad (\Rightarrow \boldsymbol{R}^{-1} = \boldsymbol{R}^T)$

* For any vector $\boldsymbol{x} \in \mathbb{R}^3$ and $\boldsymbol{R} \in \mathrm{SO}(3)$, the vector $\boldsymbol{y} = \boldsymbol{R}\boldsymbol{x}$ has the same length as $\boldsymbol{x}$ ($\|\boldsymbol{x}\| = \|\boldsymbol{R}\boldsymbol{x}\|$).

Reference Frames   **Rotation Matrices**   Angular Velocity   Exponential Coordinate Representation   Euler Angles   Unit Quaternion

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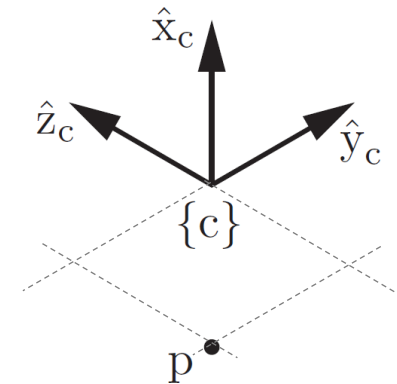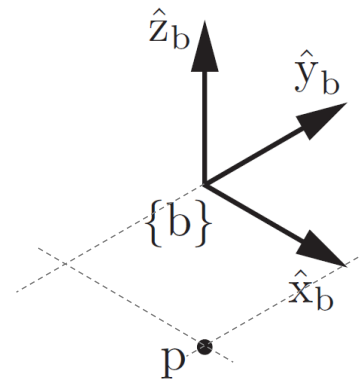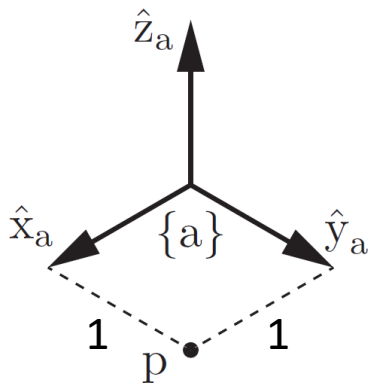
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Uses of Rotation Matrices (1)

(1) Representing orientation of a frame relative to another frame.

Notation: R_{bc} is the orientation of $\{c\}$ relative to $\{b\}$.

Example:
Find R_{ab} and R_{ac}.
(All frames have the same origin)

Uses of Rotation Matrices (2)

(2) Changing the reference frame of a <u>vector</u> or <u>frame</u>.

Subscript Cancellation Rule:

$$\boldsymbol{R}_{ab}\boldsymbol{p}_b = \boldsymbol{R}_{a\cancel{b}}\boldsymbol{p}_{\cancel{b}} = \boldsymbol{p}_a$$

$$\boldsymbol{R}_{ab}\boldsymbol{R}_{bc} = \boldsymbol{R}_{a\cancel{b}}\boldsymbol{R}_{\cancel{b}c} = \boldsymbol{R}_{ac}$$

$\boldsymbol{R}_{ab}$ can be viewed as a **<u>mathematical operator</u>** that changes the reference frame from $\{b\}$ to $\{a\}$.

Note: $\boldsymbol{R}_{bc}\boldsymbol{R}_{cb} = \boldsymbol{I}_3$ or $\boldsymbol{R}_{bc} = \boldsymbol{R}_{cb}^T = \boldsymbol{R}_{cb}^{-1}$

Example

Given $\boldsymbol{R}_1 = \boldsymbol{R}_{ab}$, $\boldsymbol{R}_2 = \boldsymbol{R}_{bc}$, and $\boldsymbol{R}_3 = \boldsymbol{R}_{ad}$, write $\boldsymbol{R}_{dc}$ in terms of $\boldsymbol{R}_1$, $\boldsymbol{R}_2$, and $\boldsymbol{R}_3$.

Given $\boldsymbol{p}_b$, what is $\boldsymbol{p}_d$ in terms of $\boldsymbol{R}_1$, $\boldsymbol{R}_2$, and $\boldsymbol{R}_3$?

Uses of Rotation Matrices (3)

(3) Rotating a <u>vector</u> or <u>frame</u> (about a unit axis $\hat{\boldsymbol{\omega}}$ by an amount θ).

$$\boldsymbol{R} = \boldsymbol{R}_{aa'} = \text{Rot}(\hat{\boldsymbol{\omega}}, \theta)$$

$\boldsymbol{R}$ can be viewed as a **mathematical operator** that rotates $\{a\}$ about a unit axis $\hat{\boldsymbol{\omega}} = (\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3)$ (expressed in $\{a\}$) by an amount θ to obtain $\{a'\}$.
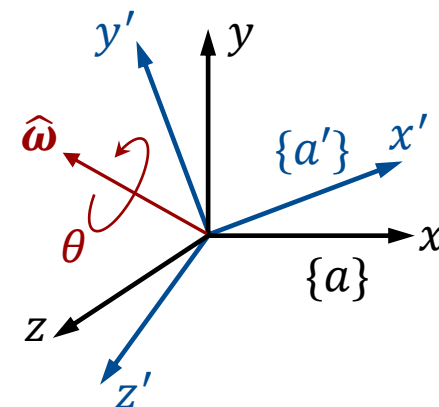
$$\text{Rot}(\hat{x}, \theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \text{Rot}(\hat{y}, \theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}, \text{Rot}(\hat{z}, \theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\text{Rot}(\hat{\boldsymbol{\omega}}, \theta) = \begin{bmatrix} c_\theta + \hat{\omega}_1^2(1 - c_\theta) & \hat{\omega}_1\hat{\omega}_2(1 - c_\theta) - \hat{\omega}_3 s_\theta & \hat{\omega}_1\hat{\omega}_3(1 - c_\theta) + \hat{\omega}_2 s_\theta \\ \hat{\omega}_1\hat{\omega}_2(1 - c_\theta) + \hat{\omega}_3 s_\theta & c_\theta + \hat{\omega}_2^2(1 - c_\theta) & \hat{\omega}_2\hat{\omega}_3(1 - c_\theta) - \hat{\omega}_1 s_\theta \\ \hat{\omega}_1\hat{\omega}_3(1 - c_\theta) - \hat{\omega}_2 s_\theta & \hat{\omega}_2\hat{\omega}_3(1 - c_\theta) + \hat{\omega}_1 s_\theta & c_\theta + \hat{\omega}_3^2(1 - c_\theta) \end{bmatrix}$$

$$s_\theta = \sin\theta, \; c_\theta = \cos\theta$$

- $\text{Rot}(\hat{\boldsymbol{\omega}}, \theta) = \text{Rot}(-\hat{\boldsymbol{\omega}}, -\theta)$

Reference Frames | Rotation Matrices | Angular Velocity | Exponential Coordinate Representation | Euler Angles | Unit Quaternion

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Uses of Rotation Matrices (3) (cont.)

- Rotation of vector $\boldsymbol{v}$ about a unit axis $\hat{\boldsymbol{\omega}}$ (expressed in the same frame) by an amount θ is vector $\boldsymbol{v}'$ expressed in the same frame:

$$\boldsymbol{v}' = \boldsymbol{R}\boldsymbol{v} = \text{Rot}(\hat{\boldsymbol{\omega}}, \theta)\boldsymbol{v}$$

- **Fixed-frame Rotation**: Rotation of frame $\{b\}$ about an axis $\hat{\boldsymbol{\omega}}$ expressed in $\{s\}$ by an amount θ is frame $\{b'\}$:

  (pre-multiplication)

$$\boldsymbol{R}_{sb'} = \text{Rot}(\hat{\boldsymbol{\omega}}, \theta)\boldsymbol{R}_{sb}$$

  Interpretation

- **Body-frame Rotation**: Rotation of frame $\{b\}$ about an axis $\hat{\boldsymbol{\omega}}$ expressed in $\{b\}$ by an amount θ is frame $\{b'\}$:

$$\boldsymbol{R}_{sb'} = \boldsymbol{R}_{sb}\text{Rot}(\hat{\boldsymbol{\omega}}, \theta)$$

  Interpretation

  (post-multiplication)

$R_{sb'} = RR_{sb}$
fixed frame rotation

$\boldsymbol{R}_{sb}$

$\boldsymbol{R} = \text{Rot}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \frac{\pi}{2}\right)$

$R_{sb'} = R_{sb}R$
body frame rotation
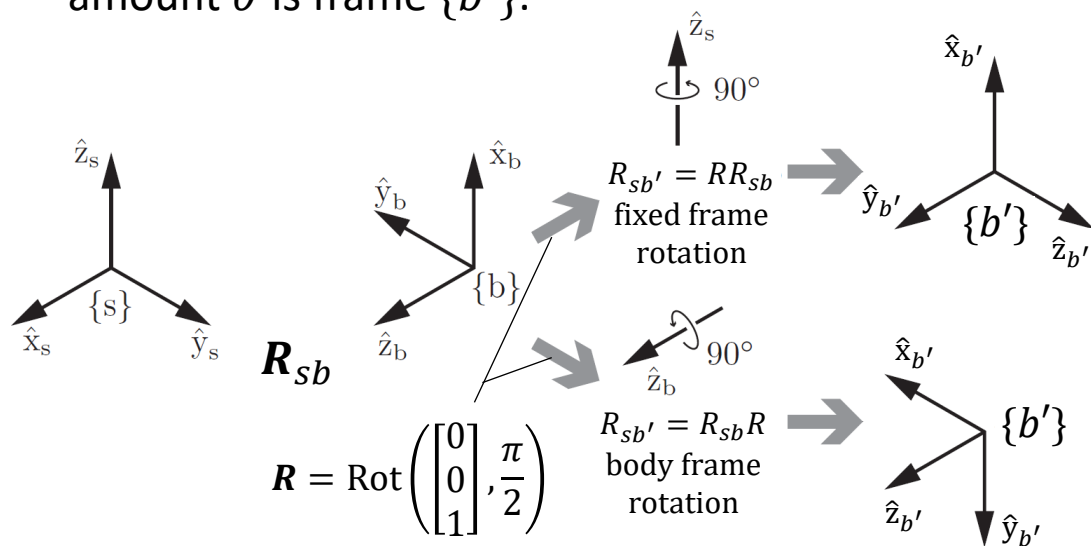
Examples
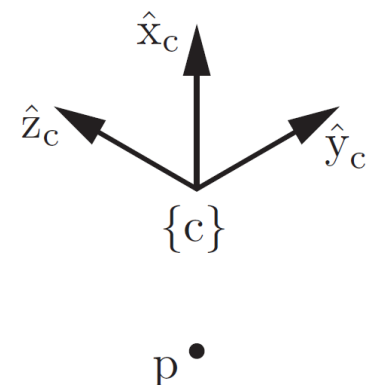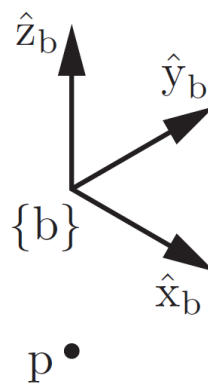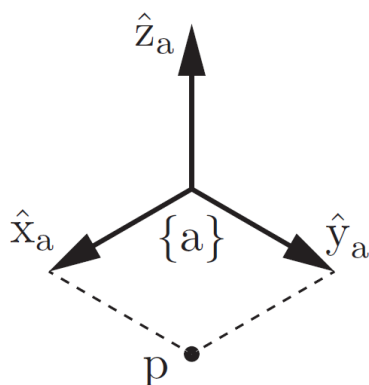
$$\boldsymbol{R} = \boldsymbol{R}_{ba} = \mathrm{Rot}(\widehat{\boldsymbol{\omega}}, \theta): \qquad \theta = \frac{\pi}{2}, \qquad \widehat{\boldsymbol{\omega}} = ?$$

$$\boldsymbol{R}_{bc'} = \boldsymbol{R}\boldsymbol{R}_{bc} = ? \qquad\qquad\qquad \boldsymbol{R}_{bc''} = \boldsymbol{R}_{bc}\boldsymbol{R} = ?$$

Angular Velocity

Reference Frames
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Rotation Matrices
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Angular Velocity
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Exponential Coordinate Representation
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Euler Angles
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Unit Quaternion
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Set of Skew-Symmetric Matrices $so(3)$

The set of all 3×3 real skew-symmetric matrices is called $so(3)$ (which is the Lie algebra of the Lie group $SO(3)$).

$$so(3) = \{\mathbf{S} \in \mathbb{R}^{3\times3} | \mathbf{S}^T = -\mathbf{S}\} \qquad so(3) \subset \mathbb{R}^{3\times3}$$

$$x \in \mathbb{R}^3 \qquad [x] \in so(3)$$

- Given any $x \in \mathbb{R}^3$ and $\boldsymbol{R} \in SO(3)$, $\boldsymbol{R}[x]\boldsymbol{R}^T = [\boldsymbol{R}x]$.

- Given $[x] \in so(3)$, $[x]^2 = xx^T - \|x\|^2 \boldsymbol{I}_3$ and $[x]^3 = -\|x\|^2 [x]$
and higher powers of $[x]$ can be calculated recursively.

Reference Frames	Rotation Matrices	Angular Velocity	Exponential Coordinate Representation	Euler Angles	Unit Quaternion
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Fixed-Frame Angular Velocity ω_s

Let's find the angular velocity $\boldsymbol{\omega} \in \mathbb{R}^3$ of a rotating body (or body frame $\{b\}$) in terms of $\boldsymbol{R}_{sb} = \boldsymbol{R}(t)$. Body Frame $\{b\}$ is instantaneously coincident with the body-attached frame.

- If $\boldsymbol{\omega}$ is expressed in $\{s\}$: $\boldsymbol{\omega} = \boldsymbol{\omega}_s = \dot{\theta}\widehat{\boldsymbol{\omega}}_s$ [1]

$\boldsymbol{R}(t) = [\hat{x}_b \quad \hat{y}_b \quad \hat{z}_b]$: $\boldsymbol{R}_{sb}$ at time t

$\dot{\boldsymbol{R}}(t) = [\dot{\hat{x}}_b \quad \dot{\hat{y}}_b \quad \dot{\hat{z}}_b]$: Time rate of change of $\boldsymbol{R}_{sb}$ at time t

$$\dot{\hat{x}}_b = \boldsymbol{\omega}_s \times \hat{x}_b$$
$$\dot{\hat{y}}_b = \boldsymbol{\omega}_s \times \hat{y}_b$$
$$\dot{\hat{z}}_b = \boldsymbol{\omega}_s \times \hat{z}_b$$
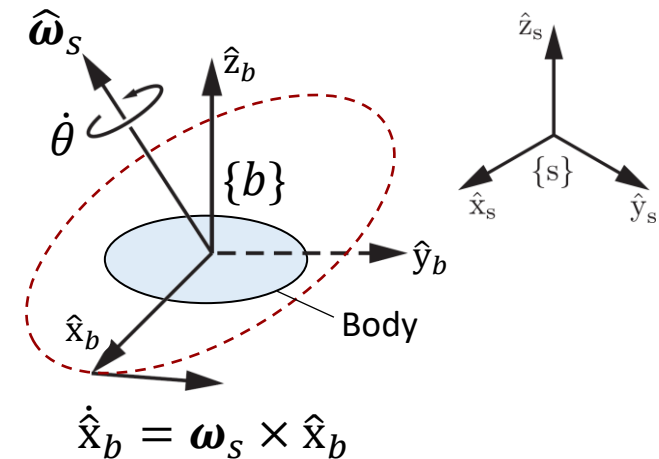
$\dot{\hat{x}}_b = \boldsymbol{\omega}_s \times \hat{x}_b$

$\boldsymbol{\omega}_s$: Fixed-frame angular velocity (angular velocity expressed in $\{s\}$)

$$\dot{\boldsymbol{R}} = [[\boldsymbol{\omega}_s]\hat{x}_b \quad [\boldsymbol{\omega}_s]\hat{y}_b \quad [\boldsymbol{\omega}_s]\hat{z}_b] = [\boldsymbol{\omega}_s]\boldsymbol{R} \qquad \Rightarrow \qquad \boxed{[\boldsymbol{\omega}_s] = \dot{\boldsymbol{R}}\boldsymbol{R}^{-1} = \dot{\boldsymbol{R}}\boldsymbol{R}^T}$$

Note: $\boldsymbol{\omega}_s$ does not depend on the choice of body frame $\{b\}$.

Reference Frames
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Rotation Matrices
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Angular Velocity
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Exponential Coordinate Representation
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Euler Angles
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Unit Quaternion
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Body-Frame Angular Velocity $\boldsymbol{\omega}_b$

- If $\boldsymbol{\omega}$ is expressed in $\{b\}$: $\boldsymbol{\omega} = \boldsymbol{\omega}_b = \dot{\theta}\widehat{\boldsymbol{\omega}}_b$ [(1)]

$\boldsymbol{\omega}_b$: Body-frame angular velocity
(angular velocity expressed in $\{b\}$)

$$\boldsymbol{\omega}_s = \boldsymbol{R}\boldsymbol{\omega}_b$$

$$\boldsymbol{\omega}_b = \boldsymbol{R}^{-1}\boldsymbol{\omega}_s = \boldsymbol{R}^T\boldsymbol{\omega}_s$$

$$\begin{aligned}[\boldsymbol{\omega}_b] &= [\boldsymbol{R}^T\boldsymbol{\omega}_s] \\ &= \boldsymbol{R}^T[\boldsymbol{\omega}_s]\boldsymbol{R} \\ &= \boldsymbol{R}^T(\dot{\boldsymbol{R}}\boldsymbol{R}^T)\boldsymbol{R} \\ &= \boldsymbol{R}^T\dot{\boldsymbol{R}} = \boldsymbol{R}^{-1}\dot{\boldsymbol{R}}\end{aligned}$$

Recall: $\boldsymbol{R}[\boldsymbol{x}]\boldsymbol{R}^T = [\boldsymbol{Rx}]$.

$$\Rightarrow \quad \boxed{[\boldsymbol{\omega}_b] = \boldsymbol{R}^{-1}\dot{\boldsymbol{R}} = \boldsymbol{R}^T\dot{\boldsymbol{R}}}$$

Note: $\boldsymbol{\omega}_b$ does not depend on the choice of fixed frame $\{s\}$.

[(1)] Any angular velocity $\boldsymbol{\omega} \in \mathbb{R}^3$ can be represented by product of a unit instantaneous axis of rotation ($\widehat{\boldsymbol{\omega}} \in S^2$) and the speed of rotation $\dot{\theta} \in \mathbb{R}$ about it, i.e., $\boldsymbol{\omega} = \|\boldsymbol{\omega}\| \, \boldsymbol{\omega}/\|\boldsymbol{\omega}\| = \dot{\theta}\widehat{\boldsymbol{\omega}}$.

Reference Frames   Rotation Matrices   **Angular Velocity**   Exponential Coordinate Representation   Euler Angles   Unit Quaternion

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Example

Find $\boldsymbol{\omega}_s$ and $\boldsymbol{\omega}_b$ for rotational motion of a one degree of freedom manipulator.

Reference Frames   Rotation Matrices   Angular Velocity   **Exponential Coordinate Representation**   Euler Angles   Unit Quaternion

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Exponential Coordinate Representation of Rotation

Reference Frames
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Rotation Matrices
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Angular Velocity
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Exponential Coordinate Representation
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Euler Angles
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Unit Quaternion
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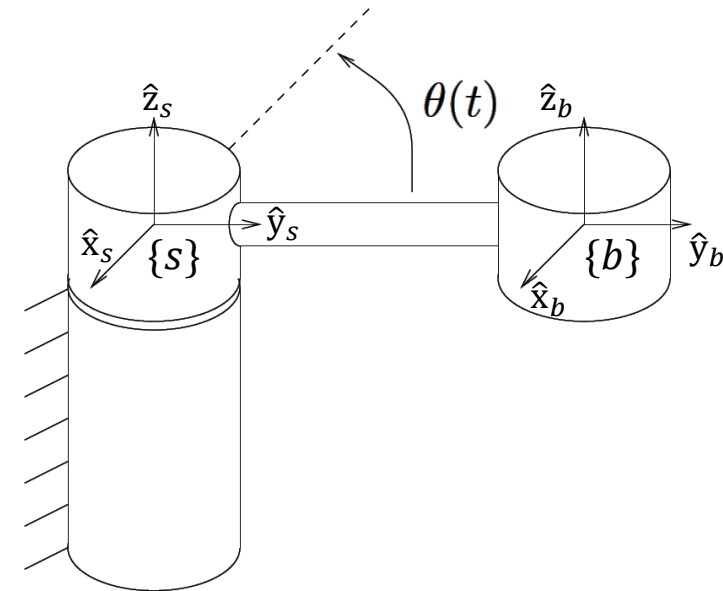
Matrix Exponential

Scalar Linear ODE:

$$\dot{x}(t) = ax(t)$$

$x(t) \in \mathbb{R}, a \in \mathbb{R}$ is constant

$x(0) = x_0$

$$x(t) = e^{at}x_0$$

Vector Linear ODE:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t)$$

$\boldsymbol{x}(t) \in \mathbb{R}^n, \boldsymbol{A} \in \mathbb{R}^{n \times n}$ is constant

$\boldsymbol{x}(0) = \boldsymbol{x}_0$

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}_0$$

Reference Frames
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Rotation Matrices
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Angular Velocity
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Exponential Coordinate Representation
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Euler Angles
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Unit Quaternion
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Properties of Matrix Exponential e^{At}

$\forall \boldsymbol{A} \in \mathbb{R}^{n \times n}, t \in \mathbb{R}$:

$$d\left(e^{\boldsymbol{A}t}\right)/dt = \boldsymbol{A}e^{\boldsymbol{A}t} = e^{\boldsymbol{A}t}\boldsymbol{A}$$

If $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$ for some $\boldsymbol{D} \in \mathbb{R}^{n \times n}$ and invertible $\boldsymbol{P} \in \mathbb{R}^{n \times n}$:

$$e^{\boldsymbol{A}t} = \boldsymbol{P}e^{\boldsymbol{D}t}\boldsymbol{P}^{-1}$$

If $\boldsymbol{D} \in \mathbb{R}^{n \times n}$ is diagonal, i.e., $\boldsymbol{D} = \text{diag}\{d_1, d_2, \ldots, d_n\}$:

$$e^{\boldsymbol{D}t} = \begin{bmatrix} e^{d_1 t} & 0 & \cdots & 0 \\ 0 & e^{d_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{d_n t} \end{bmatrix}$$

If $\boldsymbol{A}\boldsymbol{B} = \boldsymbol{B}\boldsymbol{A}$, then $e^{\boldsymbol{A}}e^{\boldsymbol{B}} = e^{\boldsymbol{A}+\boldsymbol{B}}$.

$$\left(e^{\boldsymbol{A}}\right)^{-1} = e^{-\boldsymbol{A}}$$

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Exponential Coordinates of Rotations

The vector $\boldsymbol{p}$ is rotated by an angle θ about the unit axis $\widehat{\boldsymbol{\omega}}$ to $\boldsymbol{p}'$. Thus, $\boldsymbol{p}' = \boldsymbol{R}\boldsymbol{p}$. This rotation can be also achieved by imagining that $\boldsymbol{p}$ rotates at a constant rate of $\dot{\theta} = 1$ rad/s from time $t = 0$ to $t = \theta$ (all vectors are expressed in $\{s\}$).
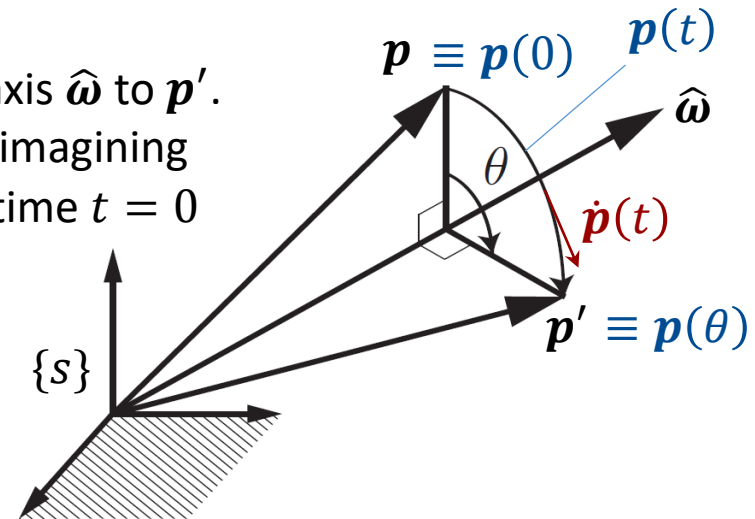
$$\dot{\boldsymbol{p}} = \dot{\theta}\widehat{\boldsymbol{\omega}} \times \boldsymbol{p}(t) = [\widehat{\boldsymbol{\omega}}]\boldsymbol{p}(t) \quad (\|\widehat{\boldsymbol{\omega}}\| = 1, \dot{\theta} = 1 \text{ rad/s})$$

$$\boldsymbol{p}(t) = e^{[\widehat{\boldsymbol{\omega}}]t}\boldsymbol{p}(0)$$

at $t = \theta$

$$\boldsymbol{p}(\theta) = e^{[\widehat{\boldsymbol{\omega}}]\theta}\boldsymbol{p}(0) \xrightarrow{\boldsymbol{p}' = \boldsymbol{R}\boldsymbol{p}} \boldsymbol{R} = e^{[\widehat{\boldsymbol{\omega}}]\theta} = \text{Rot}(\widehat{\boldsymbol{\omega}}, \theta) \in SO(3) \qquad [\widehat{\boldsymbol{\omega}}]\theta = [\widehat{\boldsymbol{\omega}}\theta] \in so(3)$$

$\Rightarrow$ Any rotation matrix $\boldsymbol{R} \in SO(3)$ can be obtained by rotating from the identity matrix $\boldsymbol{I}_3$ about a unit rotation axis $\widehat{\boldsymbol{\omega}} \in \mathbb{R}^3$ ($\|\widehat{\boldsymbol{\omega}}\| = 1$) by an angle of rotation $\theta \in \mathbb{R}$ about that axis. This motivates a three-parameter representation of a rotation $\boldsymbol{R}$ called the **exponential coordinates** as $\widehat{\boldsymbol{\omega}}\theta \in \mathbb{R}^3$ (equivalently, $\widehat{\boldsymbol{\omega}}$ and θ can be written individually as the **axis-angle representation** of a rotation).

Note: The angle θ is taken to be <u>positive</u> if the rotation is made counter-clockwise about axis $\widehat{\boldsymbol{\omega}}$.
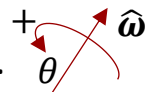
Remarks: Minimal Representation

❖ **Rotation Matrix** give an implicit redundant description of orientation; in fact, they are characterized by 9 elements which are not independent but related by 6 constraints due to the orthogonality conditions. This implies that three parameters are sufficient to describe orientation of a rigid body in space.

❖ A representation of orientation in terms of 3 independent parameters is called an **Explicit** or **Minimal Representation**. This involves **Exponential Coordinates** and **Euler Angles**. This three-parameter representation in prone to **representation singularities**.

❖ A major advantage of using rotation matrix is that it is **singularity-free** and simplifies the use of **linear algebra operations**.

Reference Frames
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Rotation Matrices
○○○○○○○○○○○○○

Angular Velocity
○○○○

Exponential Coordinate Representation
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Euler Angles
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Unit Quaternion
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Exponential Coordinates of Rotations

For any rotation matrix $\boldsymbol{R} \in SO(3)$, we can always find a unit rotation axis $\widehat{\boldsymbol{\omega}} \in \mathbb{R}^3$ ($\|\widehat{\boldsymbol{\omega}}\| = 1$) and scalar $\theta \in \mathbb{R}$ such that $\boldsymbol{R} = e^{[\widehat{\boldsymbol{\omega}}]\theta}$.

$$\text{exp:} \quad [\widehat{\boldsymbol{\omega}}]\theta \in so(3) \;\; \rightarrow \;\; \boldsymbol{R} \in SO(3) \quad : \;\; e^{[\widehat{\boldsymbol{\omega}}]\theta} = \text{Rot}(\widehat{\boldsymbol{\omega}}, \theta) = \boldsymbol{R}$$

$$\text{log:} \quad \boldsymbol{R} \in SO(3) \quad \;\; \rightarrow \;\; [\widehat{\boldsymbol{\omega}}]\theta \in so(3) \quad : \;\; \log(\boldsymbol{R}) = [\widehat{\boldsymbol{\omega}}]\theta$$

$\widehat{\boldsymbol{\omega}}\theta \in \mathbb{R}^3$: Exponential coordinates of $\boldsymbol{R} \in SO(3)$

$[\widehat{\boldsymbol{\omega}}]\theta = [\widehat{\boldsymbol{\omega}}\theta] \in so(3)$: Matrix logarithm of $\boldsymbol{R}$ (inverse of the matrix exponential)

Note: $\boldsymbol{R}$ and $\widehat{\boldsymbol{\omega}}$ have the same base.

Matrix Exponential

$$\text{exp:} \qquad [\widehat{\boldsymbol{\omega}}]\theta \in so(3) \;\rightarrow\; \boldsymbol{R} \in SO(3) \qquad : e^{[\widehat{\boldsymbol{\omega}}]\theta} = \text{Rot}(\widehat{\boldsymbol{\omega}}, \theta) = \boldsymbol{R}$$

❖ Finding $\boldsymbol{R}$ by having $\widehat{\boldsymbol{\omega}}$ and θ:

$$e^{[\widehat{\boldsymbol{\omega}}]\theta} = \boldsymbol{I} + [\widehat{\boldsymbol{\omega}}]\theta + [\widehat{\boldsymbol{\omega}}]^2 \frac{\theta^2}{2!} + [\widehat{\boldsymbol{\omega}}]^3 \frac{\theta^3}{3!} + \cdots$$

$$= \boldsymbol{I} + \underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)}_{\sin\theta} [\widehat{\boldsymbol{\omega}}] + \underbrace{\left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \cdots\right)}_{1-\cos\theta} [\widehat{\boldsymbol{\omega}}]^2$$

$$\text{Rot}(\widehat{\boldsymbol{\omega}}, \theta) = e^{[\widehat{\boldsymbol{\omega}}]\theta} = \boldsymbol{I} + \sin\theta[\widehat{\boldsymbol{\omega}}] + (1-\cos\theta)[\widehat{\boldsymbol{\omega}}]^2 \quad \text{(Rodrigues' formula for rotations)}$$

$$\text{Rot}(\widehat{\boldsymbol{\omega}}, \theta) = \begin{bmatrix} c_\theta + \widehat{\omega}_1^2(1-c_\theta) & \widehat{\omega}_1\widehat{\omega}_2(1-c_\theta) - \widehat{\omega}_3 s_\theta & \widehat{\omega}_1\widehat{\omega}_3(1-c_\theta) + \widehat{\omega}_2 s_\theta \\ \widehat{\omega}_1\widehat{\omega}_2(1-c_\theta) + \widehat{\omega}_3 s_\theta & c_\theta + \widehat{\omega}_2^2(1-c_\theta) & \widehat{\omega}_2\widehat{\omega}_3(1-c_\theta) - \widehat{\omega}_1 s_\theta \\ \widehat{\omega}_1\widehat{\omega}_3(1-c_\theta) - \widehat{\omega}_2 s_\theta & \widehat{\omega}_2\widehat{\omega}_3(1-c_\theta) + \widehat{\omega}_1 s_\theta & c_\theta + \widehat{\omega}_3^2(1-c_\theta) \end{bmatrix}$$

$$s_\theta = \sin\theta, c_\theta = \cos\theta, \quad \widehat{\boldsymbol{\omega}} = (\widehat{\omega}_1, \widehat{\omega}_2, \widehat{\omega}_3)$$

Reference Frames
○○

Rotation Matrices
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Angular Velocity
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Exponential Coordinate Representation
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Euler Angles
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Unit Quaternion
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Matrix Exponential: Remarks

- Since $\text{Rot}(\widehat{\boldsymbol{\omega}}, \theta) = \text{Rot}(-\widehat{\boldsymbol{\omega}}, -\theta)$, a rotation by $-\theta$ about $-\widehat{\boldsymbol{\omega}}$ cannot be distinguished from a rotation by θ about $\widehat{\boldsymbol{\omega}}$; hence, <u>Exponential Coordinate representation is not unique</u>.

- The inverse (or transpose) of a rotation matrix $\boldsymbol{R} = \text{Rot}(\widehat{\boldsymbol{\omega}}, \theta)$ corresponds to a rotation by the negative of the original angle, but the same axis of rotation, i.e., $\boldsymbol{R}^T = \boldsymbol{R}^{-1} = \text{Rot}(\widehat{\boldsymbol{\omega}}, -\theta)$.

- If $\boldsymbol{r}$ and $\widehat{\boldsymbol{\omega}}$ are aligned and $\boldsymbol{R} = \text{Rot}(\widehat{\boldsymbol{\omega}}, \theta)$, then, $\boldsymbol{R}\boldsymbol{r} = \boldsymbol{R}^T\boldsymbol{r} = \boldsymbol{r}$.

- For a given rotation matrix $\boldsymbol{R}_{sb}$:

  Fixed-frame Rotation is rotation by θ about a unit axis $\widehat{\boldsymbol{\omega}}_s$, expressed in fixed frame $\{s\}$ as:

  $$\boldsymbol{R}_{sb'} = \text{Rot}(\widehat{\boldsymbol{\omega}}_s, \theta)\boldsymbol{R}_{sb} = e^{[\widehat{\boldsymbol{\omega}}_s]\theta}\boldsymbol{R}_{sb}$$

  Body-frame Rotation is rotation by θ about a unit axis $\widehat{\boldsymbol{\omega}}_b$, expressed in body frame $\{b\}$ as:

  $$\boldsymbol{R}_{sb'} = \boldsymbol{R}_{sb}\text{Rot}(\widehat{\boldsymbol{\omega}}_b, \theta) = \boldsymbol{R}_{sb}e^{[\widehat{\boldsymbol{\omega}}_b]\theta}$$

  $(\widehat{\boldsymbol{\omega}}_s = \boldsymbol{R}_{sb}\widehat{\boldsymbol{\omega}}_b)$

Reference Frames
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Rotation Matrices
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Angular Velocity
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Exponential Coordinate Representation
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Euler Angles
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Unit Quaternion
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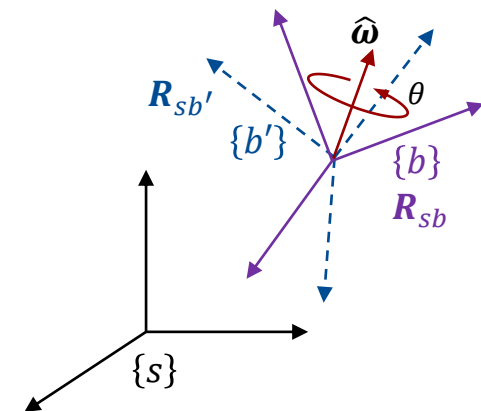
Example

The frame $\{b\}$ is obtained by a rotation from $\{s\}$ by $\theta_1 = 30°$ about $\hat{\boldsymbol{\omega}}_1 = (0, 0.866, 0.5)$.
Find the rotation matrix representation of $\{b\}$.
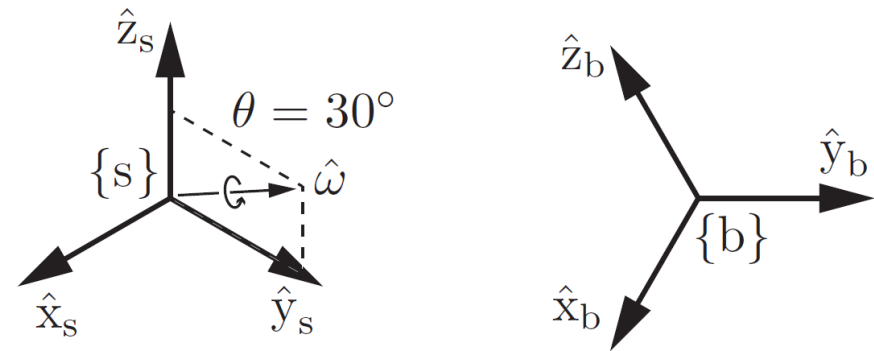
Find the new rotation matrix if $\{b\}$ is then rotated by θ_2 about
(a) an axis $\hat{\boldsymbol{\omega}}_2$ expressed in $\{s\}$.
(b) an axis $\hat{\boldsymbol{\omega}}_2$ expressed in $\{b\}$.

Matrix Logarithm

$$\log: \qquad \boldsymbol{R} \in SO(3) \qquad \rightarrow \qquad [\widehat{\boldsymbol{\omega}}]\theta \in so(3) \qquad : \qquad \log(\boldsymbol{R}) = [\widehat{\boldsymbol{\omega}}]\theta$$

❖ Finding $\widehat{\boldsymbol{\omega}}$ and $\theta \in [0, \pi]$ by having $\boldsymbol{R}$:

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c_\theta + \widehat{\omega}_1^2(1 - c_\theta) & \widehat{\omega}_1\widehat{\omega}_2(1 - c_\theta) - \widehat{\omega}_3 s_\theta & \widehat{\omega}_1\widehat{\omega}_3(1 - c_\theta) + \widehat{\omega}_2 s_\theta \\ \widehat{\omega}_1\widehat{\omega}_2(1 - c_\theta) + \widehat{\omega}_3 s_\theta & c_\theta + \widehat{\omega}_2^2(1 - c_\theta) & \widehat{\omega}_2\widehat{\omega}_3(1 - c_\theta) - \widehat{\omega}_1 s_\theta \\ \widehat{\omega}_1\widehat{\omega}_3(1 - c_\theta) - \widehat{\omega}_2 s_\theta & \widehat{\omega}_2\widehat{\omega}_3(1 - c_\theta) + \widehat{\omega}_1 s_\theta & c_\theta + \widehat{\omega}_3^2(1 - c_\theta) \end{bmatrix}$$

By inspection:

- $\operatorname{tr} \boldsymbol{R} = r_{11} + r_{22} + r_{33} = 1 + 2\cos\theta$

- $\dfrac{1}{2\sin\theta}\left(\boldsymbol{R} - \boldsymbol{R}^{\mathrm{T}}\right) = \begin{bmatrix} 0 & -\widehat{\omega}_3 & \widehat{\omega}_2 \\ \widehat{\omega}_3 & 0 & -\widehat{\omega}_1 \\ -\widehat{\omega}_2 & \widehat{\omega}_1 & 0 \end{bmatrix} = [\widehat{\boldsymbol{\omega}}]$

- $\boldsymbol{R}|_{\theta=0} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boldsymbol{I}_3$ • $\boldsymbol{R}|_{\theta=\pi} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} -1 + 2\widehat{\omega}_1^2 & 2\widehat{\omega}_1\widehat{\omega}_2 & 2\widehat{\omega}_1\widehat{\omega}_3 \\ 2\widehat{\omega}_1\widehat{\omega}_2 & -1 + 2\widehat{\omega}_2^2 & 2\widehat{\omega}_2\widehat{\omega}_3 \\ 2\widehat{\omega}_1\widehat{\omega}_3 & 2\widehat{\omega}_2\widehat{\omega}_3 & -1 + 2\widehat{\omega}_3^2 \end{bmatrix}$

Therefore, we can propose an algorithm to determine $\widehat{\boldsymbol{\omega}}$ and θ.

Reference Frames
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Rotation Matrices
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Angular Velocity
○○○○

Exponential Coordinate Representation
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Euler Angles
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Unit Quaternion
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Matrix Logarithm: Algorithm

(a) If $\text{tr}\boldsymbol{R} = 3$ or $\boldsymbol{R} = \boldsymbol{I}_3$ (null rotation), then $\theta = 0$ and $\widehat{\boldsymbol{\omega}}$ is undefined/arbitrary (singularity).

(b) If $\text{tr}\boldsymbol{R} = -1$, then $\theta = \pi$ and $\widehat{\boldsymbol{\omega}}$ is equal to any of the three vectors that is a feasible solution:

$$\widehat{\boldsymbol{\omega}} = \frac{1}{\sqrt{2(1+r_{11})}}\begin{bmatrix} 1+r_{11} \\ r_{21} \\ r_{31} \end{bmatrix} \quad \text{or} \quad \widehat{\boldsymbol{\omega}} = \frac{1}{\sqrt{2(1+r_{22})}}\begin{bmatrix} r_{12} \\ 1+r_{22} \\ r_{32} \end{bmatrix} \quad \text{or} \quad \widehat{\boldsymbol{\omega}} = \frac{1}{\sqrt{2(1+r_{33})}}\begin{bmatrix} r_{13} \\ r_{23} \\ 1+r_{33} \end{bmatrix}$$

(Note: In this case, if $\widehat{\boldsymbol{\omega}}$ is a solution, then so is $-\widehat{\boldsymbol{\omega}}$)

(c) Otherwise, $\qquad \theta = \cos^{-1}\left(\frac{1}{2}(\text{tr}\,\boldsymbol{R} - 1)\right) = \cos^{-1}\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right) \in (0, \pi)$

$$[\widehat{\omega}] = \frac{1}{2\sin\theta}(\boldsymbol{R} - \boldsymbol{R}^{\text{T}}) \quad \Rightarrow \quad \widehat{\boldsymbol{\omega}} = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

Reference Frames   Rotation Matrices   Angular Velocity   Exponential Coordinate Representation   **Euler Angles**   Unit Quaternion

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Euler Angles

Reference Frames ○○ | Rotation Matrices ○○○○○○○○○○○○ | Angular Velocity ○○○○ | Exponential Coordinate Representation ○○○○○○○○○○ | **Euler Angles** ●○○○○ | Unit Quaternion ○○○○

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Euler Angles

Another minimal representation of orientation can be obtained by using a set of three angles (α, β, γ). It composes <u>a suitable sequence of three elementary rotations</u>, each about one of the coordinate axes of fixed frame $\{s\}$ or body/current frame $\{b\}$, while guaranteeing that <u>two successive rotations are not made about parallel axes</u>.

This implies that 12 (3x2x2) distinct sets (triplet) of Euler angles are allowed out of all 27 (3x3x3) possible combinations.

Z-X-Z, X-Y-X, Y-Z-Y, Z-Y-Z, X-Z-X, Y-X-Y, X-Y-Z, Y-Z-X, Z-X-Y, X-Z-Y, Z-Y-X, Y-X-Z
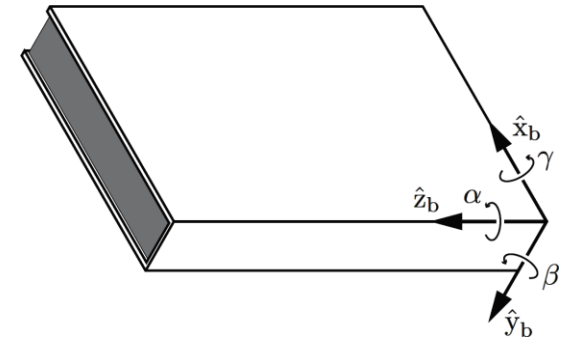
Two Examples:
- ZYX Euler angles (with rotations about the body/current frame $\{b\}$).
- XYZ Euler angles (with rotations about the fixed frame $\{s\}$). This is also called **Roll–Pitch–Yaw** (**RPY**) angles.

Euler Angles ZYX
(about the body/current frame)

ZYX Euler angles (with rotations about the body/current frame $\{b\}$):

- Rotation by α about the body $\hat{\boldsymbol{z}}_b$,
- then by β about the body $\hat{\boldsymbol{y}}_b'$, and
- finally, by γ about the body $\hat{\boldsymbol{x}}_b''$.

$$\boldsymbol{R}(\alpha, \beta, \gamma) = \boldsymbol{I}_3 \mathrm{Rot}(\hat{\boldsymbol{z}}_b, \alpha) \mathrm{Rot}(\hat{\boldsymbol{y}}_b', \beta) \mathrm{Rot}(\hat{\boldsymbol{x}}_b'', \gamma)$$

$$\mathrm{Rot}(\hat{\boldsymbol{x}}, \gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}, \mathrm{Rot}(\hat{\boldsymbol{y}}, \beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix},$$

$$\mathrm{Rot}(\hat{\boldsymbol{z}}, \alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\boldsymbol{R}(\alpha, \beta, \gamma) = \begin{bmatrix} c_\alpha c_\beta & c_\alpha s_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta c_\gamma + s_\alpha s_\gamma \\ s_\alpha c_\beta & s_\alpha s_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta c_\gamma - c_\alpha s_\gamma \\ -s_\beta & c_\beta s_\gamma & c_\beta c_\gamma \end{bmatrix}$$

Reference Frames ○○ | Rotation Matrices ○○○○○○○○○○○○○ | Angular Velocity ○○○○ | Exponential Coordinate Representation ○○○○○○○○○○ | **Euler Angles** ○○●○○ | Unit Quaternion ○○○○

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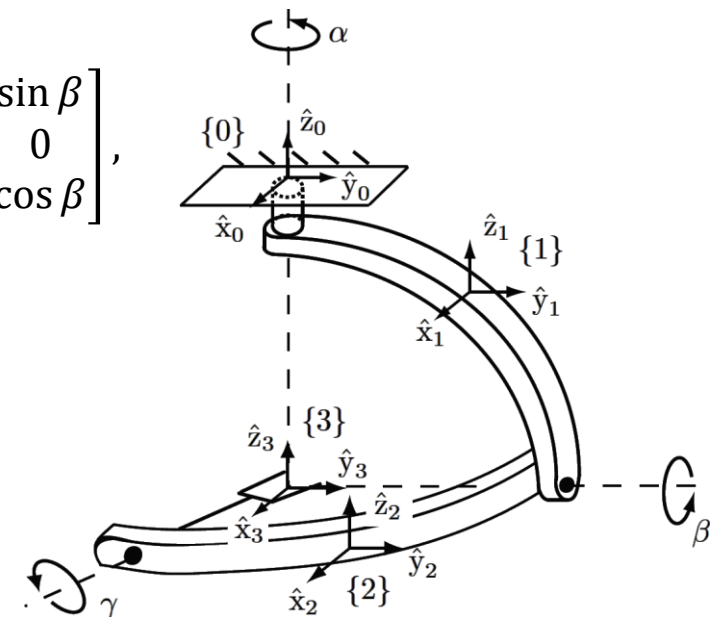
Euler Angles ZYX
(about the body/current frame)

Finding (α, β, γ) for any given rotation matrix $\boldsymbol{R} \in SO(3)$:

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c_\alpha c_\beta & c_\alpha s_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta c_\gamma + s_\alpha s_\gamma \\ s_\alpha c_\beta & s_\alpha s_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta c_\gamma - c_\alpha s_\gamma \\ -s_\beta & c_\beta s_\gamma & c_\beta c_\gamma \end{bmatrix}$$
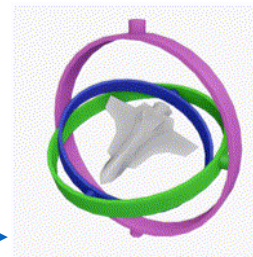
- If $r_{31} \neq \pm 1$ (i.e., when $\beta \in (-\pi/2, \pi/2)$):

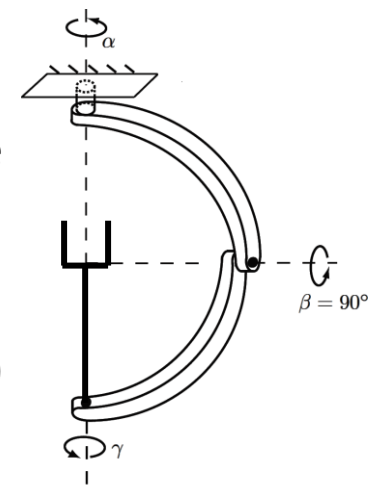$$\beta = \text{atan} 2\left(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}\right)$$

$$\alpha = \text{atan} 2(r_{21}, r_{11})$$

$$\gamma = \text{atan} 2(r_{32}, r_{33})$$

- If $r_{31} = -1$, then $\beta = \pi/2$, and if $r_{31} = 1$, then $\beta = -\pi/2$. In these singular cases, $\hat{z}_b$ and $\hat{x}_b$ axes are parallel, and it is possible to determine only the sum or difference of α and γ.

Singularity of the Euler angles:

(Gimbal lock)

▶   ▶

$\beta = 90°$

Reference Frames ○○ | Rotation Matrices ○○○○○○○○○○○○○ | Angular Velocity ○○○○ | Exponential Coordinate Representation ○○○○○○○○○○ | Euler Angles ○○○●○ | Unit Quaternion ○○○○

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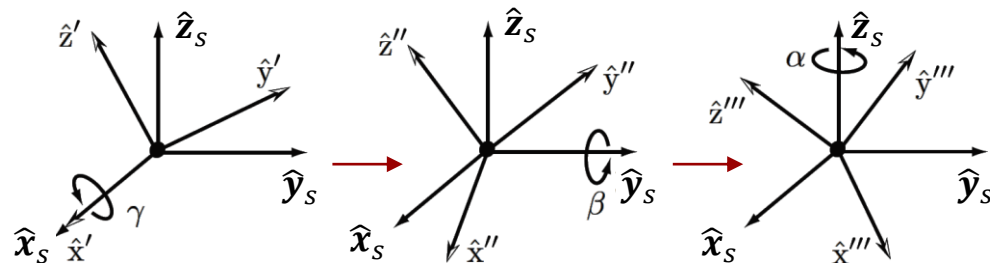# Roll–Pitch–Yaw or RPY Angles (XYZ)
(about the fixed frame)

XYZ Euler angles (with rotations about the fixed frame $\{s\}$):

- Rotation by γ about the body $\hat{\boldsymbol{x}}_s$,
- then by β about the body $\hat{\boldsymbol{y}}_s$, and
- finally, by α about the body $\hat{\boldsymbol{z}}_s$.

$$\boldsymbol{R}(\alpha, \beta, \gamma) = \mathrm{Rot}(\hat{\boldsymbol{z}}_s, \alpha)\,\mathrm{Rot}(\hat{\boldsymbol{y}}_s, \beta)\,\mathrm{Rot}(\hat{\boldsymbol{x}}_s, \gamma)\,\boldsymbol{I}_3$$

$$\boldsymbol{R}(\alpha, \beta, \gamma) = \begin{bmatrix} c_\alpha c_\beta & c_\alpha s_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta c_\gamma + s_\alpha s_\gamma \\ s_\alpha c_\beta & s_\alpha s_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta c_\gamma - c_\alpha s_\gamma \\ -s_\beta & c_\beta s_\gamma & c_\beta c_\gamma \end{bmatrix}$$

This product of three rotations (i.e., XYZ Euler angles with rotations about the fixed frame $\{s\}$) is the same as that for the ZYX Euler angles with rotations about the body/current frame $\{b\}$, i.e., <u>the same product of three rotations admits two different physical interpretations</u>.

Reference Frames   Rotation Matrices   Angular Velocity   Exponential Coordinate Representation   **Euler Angles**   Unit Quaternion

OO   OOOOOOOOOOOO   OOOO   OOOOOOOOOO   OOOO●   OOOO

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Successive Rotations about Axes of Fixed & Body/Current Frames

Body-frame rotation:

$$\boldsymbol{R} = \boldsymbol{I}_3 \boldsymbol{R}(y, \pi/2)\boldsymbol{R}(z', \pi/2)$$

Fixed-frame rotation:

$$\boldsymbol{R} = \boldsymbol{R}(y, \pi/2)\boldsymbol{R}(z, \pi/2)\boldsymbol{I}_3$$

Reference Frames
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Rotation Matrices
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Angular Velocity
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Exponential Coordinate Representation
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Euler Angles
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Unit Quaternion
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Unit Quaternion

Reference Frames   Rotation Matrices   Angular Velocity   Exponential Coordinate Representation   Euler Angles   **Unit Quaternion**
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Unit Quaternion

Unit Quaternion (a.k.a. **Euler Parameters**) is a nonminimal four-parameter representation of rotation that alleviates the exponential coordinates singularity (division by $\sin\theta$) and Euler angle singularity (Gimbal lock), but at the cost of four variables subject to one constraint in the representation.

Let $\boldsymbol{R} \in SO(3)$ have the exponential coordinate representation $\hat{\boldsymbol{\omega}}\theta$, i.e., $\boldsymbol{R} = e^{[\hat{\boldsymbol{\omega}}]\theta}$, where $\|\hat{\boldsymbol{\omega}}\| = 1$ and $\theta = [0, \pi]$. The Unit Quaternion is defined as

scalar part

$$\boldsymbol{q} = \begin{bmatrix} q_0 \\ \boldsymbol{q}_v \end{bmatrix} = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} \cos(\theta/2) \\ \hat{\boldsymbol{\omega}}\sin(\theta/2) \end{bmatrix} \in S^3 \qquad \|\boldsymbol{q}\| = 1$$

vector part

- $\boldsymbol{q} = (q_0, \boldsymbol{q}_v) = (q_0, q_1, q_2, q_3)$ is interpreted as a rotation about the unit axis, in the direction of (q_1, q_2, q_3) by an angle $\theta = 2\cos^{-1} q_0$.
- If $0 \leq \theta \leq \pi$, then $q_0 \geq 0$, and if $\pi \leq \theta \leq 2\pi$, then $q_0 \leq 0$.

Reference Frames ○○ | Rotation Matrices ○○○○○○○○○○○○○ | Angular Velocity ○○○○ | Exponential Coordinate Representation ○○○○○○○○○○ | Euler Angles ○○○○○ | Unit Quaternion ○●○○

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Unit Quaternion: Finding R by Having q

The rotation matrix $\boldsymbol{R}$ corresponding to a given unit quaternion $\boldsymbol{q} = (q_0, \boldsymbol{q}_v) = (q_0, q_1, q_2, q_3) = \left(\cos(\theta/2), \widehat{\boldsymbol{\omega}} \sin\left(\frac{\theta}{2}\right)\right)$ where $\|\boldsymbol{q}\| = q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$:

$$\boldsymbol{R} = \begin{bmatrix} c_\theta + \widehat{\omega}_1^2(1 - c_\theta) & \widehat{\omega}_1\widehat{\omega}_2(1 - c_\theta) - \widehat{\omega}_3 s_\theta & \widehat{\omega}_1\widehat{\omega}_3(1 - c_\theta) + \widehat{\omega}_2 s_\theta \\ \widehat{\omega}_1\widehat{\omega}_2(1 - c_\theta) + \widehat{\omega}_3 s_\theta & c_\theta + \widehat{\omega}_2^2(1 - c_\theta) & \widehat{\omega}_2\widehat{\omega}_3(1 - c_\theta) - \widehat{\omega}_1 s_\theta \\ \widehat{\omega}_1\widehat{\omega}_3(1 - c_\theta) - \widehat{\omega}_2 s_\theta & \widehat{\omega}_2\widehat{\omega}_3(1 - c_\theta) + \widehat{\omega}_1 s_\theta & c_\theta + \widehat{\omega}_3^2(1 - c_\theta) \end{bmatrix}$$

$$= \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1 q_2 - q_0 q_3) & 2(q_0 q_2 + q_1 q_3) \\ 2(q_0 q_3 + q_1 q_2) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2 q_3 - q_0 q_1) \\ 2(q_1 q_3 - q_0 q_2) & 2(q_0 q_1 + q_2 q_3) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}$$

$$= \begin{bmatrix} 2(q_0^2 + q_1^2) - 1 & 2(q_1 q_2 - q_0 q_3) & 2(q_1 q_3 + q_0 q_2) \\ 2(q_1 q_2 + q_0 q_3) & 2(q_0^2 + q_2^2) - 1 & 2(q_2 q_3 - q_0 q_1) \\ 2(q_1 q_3 - q_0 q_2) & 2(q_2 q_3 + q_0 q_1) & 2(q_0^2 + q_3^2) - 1 \end{bmatrix}$$

$$= (2q_0^2 - 1)\boldsymbol{I}_3 + 2\boldsymbol{q}_v\boldsymbol{q}_v^T + 2q_0[\boldsymbol{q}_v]$$

Reference Frames
OO

Rotation Matrices
OOOOOOOOOOOO

Angular Velocity
OOOO

Exponential Coordinate Representation
OOOOOOOOOO

Euler Angles
OOOOO

Unit Quaternion
OOOO

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Unit Quaternion: Finding q by Having R

The unit quaternion $\boldsymbol{q}$ corresponding to a given rotation matrix $\boldsymbol{R}$:
$\qquad \boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$

$$q_0 = \frac{1}{2}\sqrt{1 + r_{11} + r_{22} + r_{33}} = \frac{1}{2}\sqrt{1 + \operatorname{tr}\boldsymbol{R}}$$

$$\boldsymbol{q}_v = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \frac{1}{4q_0}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - 2_{12} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \operatorname{sgn}(r_{32} - r_{23})\sqrt{r_{11} - r_{22} - r_{33} + 1} \\ \operatorname{sgn}(r_{13} - r_{31})\sqrt{r_{22} - r_{33} - r_{11} + 1} \\ \operatorname{sgn}(r_{21} - r_{12})\sqrt{r_{33} - r_{11} - r_{22} + 1} \end{bmatrix}$$

Notes:

- It has been implicitly assumed $q_0 \geq 0$; this corresponds to an angle $\theta \in [-\pi, \pi]$, and thus, any rotation can be described.
- For every rotation matrix $\boldsymbol{R}$ there exists two unit-quaternion representations $\pm\boldsymbol{q}$ that are **antipodal** to each other, i.e., $+\boldsymbol{q}$ and $-\boldsymbol{q}$ represent the same rotation $\boldsymbol{R}$.
- Unlike the exponential coordinate representation, no singularity occurs in inverse solution.
- A rotation by $-\theta$ about $-\widehat{\boldsymbol{\omega}}$ gives the same quaternion $\boldsymbol{q}$ as that associated with a rotation by θ about $\widehat{\boldsymbol{\omega}}$.
- If the unit quaternion $\boldsymbol{q} = (q_0, \boldsymbol{q}_v)$ corresponds to a rotation matrix $\boldsymbol{R}$, the unit quaternion extracted from $\boldsymbol{R}^{-1} = \boldsymbol{R}^T$ is denoted as $\boldsymbol{q}^{-1}$, and can be computed as $\boldsymbol{q}^{-1} = (q_0, -\boldsymbol{q}_v)$.

Reference Frames ○○

Rotation Matrices ○○○○○○○○○○○○○○

Angular Velocity ○○○○

Exponential Coordinate Representation ○○○○○○○○○○

Euler Angles ○○○○○

Unit Quaternion ○○○●

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Unit Quaternion: Remarks

- Let $\boldsymbol{p} = (p_0, \boldsymbol{p}_v) = (p_0, p_1, p_2, p_3)$ and $\boldsymbol{q} = (q_0, \boldsymbol{q}_v) = (q_0, q_1, q_2, q_3)$ denote the quaternions corresponding to the rotation matrices $\boldsymbol{R}_p$ and $\boldsymbol{R}_q$, respectively. The quaternion corresponding to the product $\boldsymbol{R}_n = \boldsymbol{R}_p \boldsymbol{R}_q$ is given by

$$\boldsymbol{n} = \boldsymbol{p} * \boldsymbol{q} = \begin{bmatrix} n_0 \\ n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} p_0 q_0 - p_1 q_1 - p_2 q_2 - p_3 q_3 \\ p_1 q_0 + p_0 q_1 - p_3 q_2 + p_2 q_3 \\ p_2 q_0 + p_0 q_2 + p_3 q_1 - p_1 q_3 \\ p_3 q_0 + p_0 q_3 - p_2 q_1 + p_1 q_2 \end{bmatrix}$$

quaternion product operator

$$= (p_0 q_0 - \boldsymbol{p}_v^T \boldsymbol{q}_v, p_0 \boldsymbol{q}_v + q_0 \boldsymbol{p}_v + \boldsymbol{p}_v \times \boldsymbol{q}_v)$$

- $\boldsymbol{q} * \boldsymbol{q}^{-1} = (1, \boldsymbol{0})$ which is the identity element.

- The rotation of a point or vector $\boldsymbol{p} \in \mathbb{R}^3$ by the angle θ about a unit axis $\hat{\boldsymbol{\omega}}$ is $\boldsymbol{p}' = \text{Rot}(\hat{\boldsymbol{\omega}}, \theta) \, \boldsymbol{p}$ which can be also determined using unit quaternions as $\boldsymbol{q}_{p'} = \boldsymbol{q} * \boldsymbol{q}_p * \boldsymbol{q}^{-1}$ where $\boldsymbol{q} = (q_0, \boldsymbol{q}_v)$ is unit quaternion representation of $\text{Rot}(\hat{\boldsymbol{\omega}}, \theta)$, $\boldsymbol{q}^{-1} = (q_0, -\boldsymbol{q}_v)$, $\boldsymbol{q}_p = (0, \boldsymbol{p})$, and $\boldsymbol{q}_{p'} = (0, \boldsymbol{p}')$.

$$\boldsymbol{p}' = \text{Rot}(\hat{\boldsymbol{\omega}}, \theta) \, \boldsymbol{p} \quad \leftrightarrow \quad \boldsymbol{q}_{p'} = \boldsymbol{q} * \boldsymbol{q}_p * \boldsymbol{q}^{-1}$$