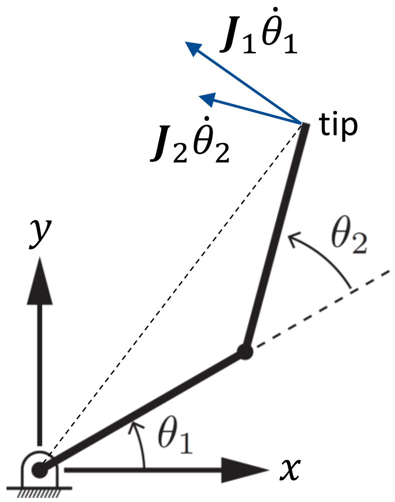
Ch6: Velocity Kinematics and Statics

Geometric Jacobian

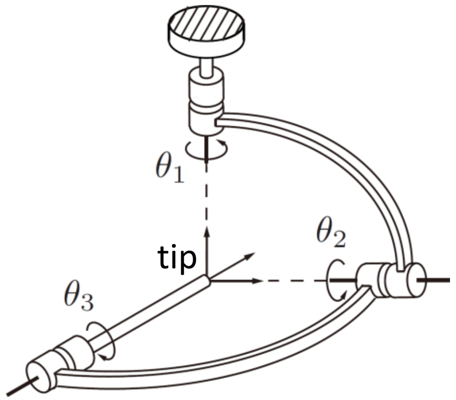
Manipulator Jacobian

In a 2R planar robot, we saw that $\boldsymbol{v}_{\text{tip}}$ is the linear velocity of the end-effector frame

$$\boldsymbol{v}_{\text{tip}} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \boldsymbol{J}(\theta_1, \theta_2) \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = [\boldsymbol{J}_1 \quad \boldsymbol{J}_2] \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \boldsymbol{J}_1\dot{\theta}_1 + \boldsymbol{J}_2\dot{\theta}_2$$



In a pure orienting devices such as a wrist, $\boldsymbol{v}_{\text{tip}}$ is the angular velocity of the end-effector frame.



- Thus, $\boldsymbol{v}_{\text{tip}}$ determine the specific form of the Jacobian.

Space and Body Manipulator Jacobians

Let's assume that the configuration of the end-effector is expressed as $\boldsymbol{T}_{sb} = \boldsymbol{T} \in SE(3)$ and its velocity is expressed as a twist $\boldsymbol{\mathcal{V}} \in \mathbb{R}^6$ in the fixed base frame $\{s\}$ or the end-effector body frame $\{b\}$.

❖ The **Jacobian** is derived based on the following general idea:
Given the configuration $\boldsymbol{\theta} \in \mathbb{R}^n$ of the robot, $\boldsymbol{J}_i(\boldsymbol{\theta}) \in \mathbb{R}^6$, which is column i of $\boldsymbol{J}(\boldsymbol{\theta}) \in \mathbb{R}^{6\times n}$, is the twist $\boldsymbol{\mathcal{V}}$ when the robot is in an arbitrary configuration $\boldsymbol{\theta}$ (not in zero configuration $\boldsymbol{\theta} = \boldsymbol{0}$), $\dot{\theta}_i = 1$, and all other joint velocities are zero.

$$\boldsymbol{\mathcal{V}} = \boldsymbol{J}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} = [\boldsymbol{J}_1 \quad \boldsymbol{J}_2 \quad \cdots \quad \boldsymbol{J}_n]\dot{\boldsymbol{\theta}}$$

$\dot{\boldsymbol{\theta}}$: joint velocities

(Velocity or Differential Kinematics Equation is a linear map in a given configuration)

- If each column $\boldsymbol{J}_i(\boldsymbol{\theta})$ is expressed in the fixed space frame $\{s\}$: $\Rightarrow$ Space Jacobian $\boldsymbol{\mathcal{V}}_s = \boldsymbol{J}_s(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$
- If each column $\boldsymbol{J}_i(\boldsymbol{\theta})$ is expressed in the end-effector frame $\{b\}$: $\Rightarrow$ Body Jacobian $\boldsymbol{\mathcal{V}}_b = \boldsymbol{J}_b(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$

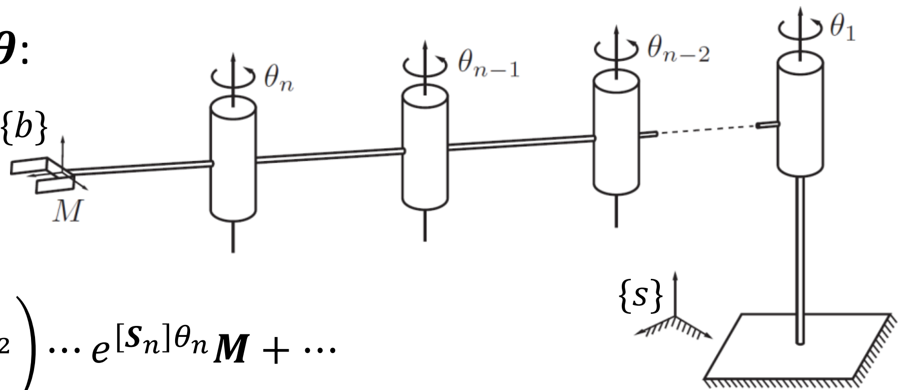
Geometric Jacobians

Space Jacobian

Consider an n-link open chain as configuration $\boldsymbol{\theta}$:

$$\boldsymbol{T}(\boldsymbol{\theta}) = e^{[\boldsymbol{S}_1]\theta_1} e^{[\boldsymbol{S}_2]\theta_2} \dots e^{[\boldsymbol{S}_n]\theta_n} \boldsymbol{M}$$ forward kinematics

$$[\boldsymbol{\mathcal{V}}_s] = \dot{\boldsymbol{T}}\boldsymbol{T}^{-1} \qquad (\boldsymbol{T} = \boldsymbol{T}_{sb})$$



$$\dot{\boldsymbol{T}} = \left(\frac{d}{dt} e^{[\boldsymbol{S}_1]\theta_1}\right) \cdots e^{[\boldsymbol{S}_n]\theta_n} \boldsymbol{M} + e^{[\boldsymbol{S}_1]\theta_1} \left(\frac{d}{dt} e^{[\boldsymbol{S}_2]\theta_2}\right) \cdots e^{[\boldsymbol{S}_n]\theta_n} \boldsymbol{M} + \cdots$$

$$= [\boldsymbol{S}_1]\dot{\theta}_1 e^{[\boldsymbol{S}_1]\theta_1} \dots e^{[\boldsymbol{S}_n]\theta_n} \boldsymbol{M} + e^{[\boldsymbol{S}_1]\theta_1} [\boldsymbol{S}_2]\dot{\theta}_2 e^{[\boldsymbol{S}_2]\theta_2} \dots e^{[\boldsymbol{S}_n]\theta_n} \boldsymbol{M} + \cdots$$

$$\boldsymbol{T}^{-1} = \boldsymbol{M}^{-1} e^{-[\boldsymbol{S}_n]\theta_n} \dots e^{-[\boldsymbol{S}_1]\theta_1}$$

$$[\boldsymbol{\mathcal{V}}_s] = [\boldsymbol{S}_1]\dot{\theta}_1 + e^{[\boldsymbol{S}_1]\theta_1} [\boldsymbol{S}_2] e^{-[\boldsymbol{S}_1]\theta_1} \dot{\theta}_2 + e^{[\boldsymbol{S}_1]\theta_1} e^{[\boldsymbol{S}_2]\theta_2} [\boldsymbol{S}_3] e^{-[\boldsymbol{S}_2]\theta_2} e^{-[\boldsymbol{S}_1]\theta_1} \dot{\theta}_3 + \cdots$$

$$\boldsymbol{A}[\boldsymbol{S}_i]\boldsymbol{A}^{-1} = [[\mathrm{Ad}_{\boldsymbol{A}}]\boldsymbol{S}_i] \qquad \boldsymbol{A} \in SE(3)$$

$$\boldsymbol{\mathcal{V}}_s = \underbrace{\boldsymbol{S}_1}_{\boldsymbol{J}_{s1}} \dot{\theta}_1 + \underbrace{\left[\mathrm{Ad}_{e^{[\boldsymbol{S}_1]\theta_1}}\right]\boldsymbol{S}_2}_{\boldsymbol{J}_{s2}} \dot{\theta}_2 + \underbrace{\left[\mathrm{Ad}_{e^{[\boldsymbol{S}_1]\theta_1} e^{[\boldsymbol{S}_2]\theta_2}}\right]\boldsymbol{S}_3}_{\boldsymbol{J}_{s3}} \dot{\theta}_3 + \cdots$$

$$\boldsymbol{\mathcal{V}}_s = \boldsymbol{J}_{s1}\dot{\theta}_1 + \boldsymbol{J}_{s2}(\theta_1)\dot{\theta}_2 + \cdots + \boldsymbol{J}_{sn}(\theta_1, \dots, \theta_{n-1})\dot{\theta}_n$$

Space Jacobian (cont.)

$$\boldsymbol{\mathcal{V}}_s = [\boldsymbol{J}_{s_1} \quad \boldsymbol{J}_{s2}(\theta_1) \quad \cdots \quad \boldsymbol{J}_{s_n}(\theta_1, \dots, \theta_{n-1})] \begin{bmatrix} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_n \end{bmatrix} = \boldsymbol{J}_s(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$$

The **space Jacobian** $\boldsymbol{J}_s(\boldsymbol{\theta}) \in \mathbb{R}^{6\times n}$ relates the joint rate vector $\dot{\boldsymbol{\theta}} \in \mathbb{R}^n$ to the spatial twist $\boldsymbol{\mathcal{V}}_s$. The ith column of $\boldsymbol{J}_s(\boldsymbol{\theta})$ is

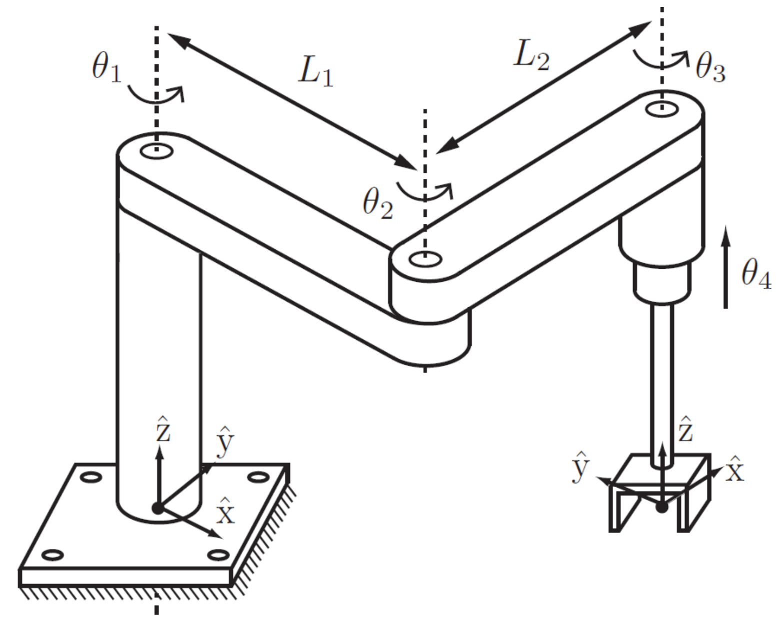
$$\boldsymbol{J}_{si}(\boldsymbol{\theta}) = \begin{bmatrix} \boldsymbol{\omega}_{si}(\boldsymbol{\theta}) \\ \boldsymbol{v}_{si}(\boldsymbol{\theta}) \end{bmatrix} = \left[\mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1} \dots e^{[\mathcal{S}_{i-1}]\theta_{i-1}}}\right] \boldsymbol{\mathcal{S}}_i \qquad \begin{matrix} \boldsymbol{J}_{s1} = \boldsymbol{\mathcal{S}}_1 \\ i = 2, \dots, n \end{matrix}$$

Screw axis describing the ith joint axis (expressed in the fixed space frame $\{s\}$) after the joints $1, \dots, i-1$ move from their zero position to the current values $\theta_1, \dots, \theta_{i-1}$.

Screw axis describing the ith joint axis (expressed in the fixed space frame $\{s\}$) when the robot in its zero/home configuration $\boldsymbol{\theta} = \boldsymbol{0}$.

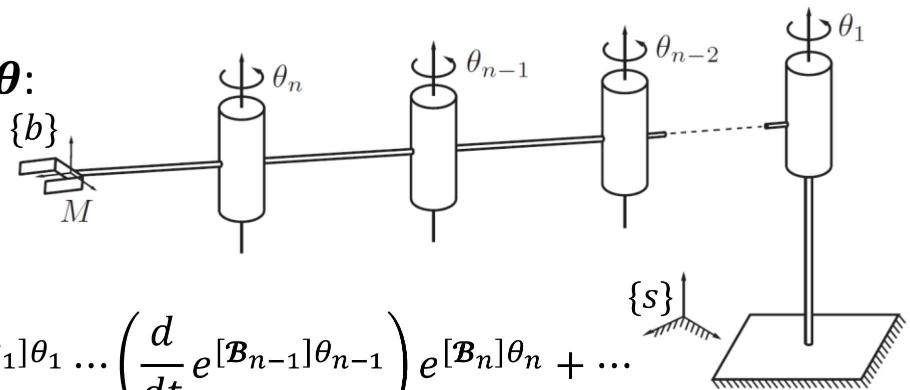
Note: The space Jacobian $\boldsymbol{J}_s$ is independent of the choice of the end-effector frame $\{b\}$.
Note: $\boldsymbol{J}_{si}$ is determined in the same way as the joint screw axis $\boldsymbol{\mathcal{S}}_i$, except that $\boldsymbol{J}_{si}$ is determined for an arbitrary $\boldsymbol{\theta}$ rather than $\boldsymbol{\theta} = \boldsymbol{0}$.

Example: Space Jacobian of a Spatial RRRP Robot



Body Jacobian

Consider an n-link open chain as configuration $\boldsymbol{\theta}$:



$$\boldsymbol{T}(\boldsymbol{\theta}) = \boldsymbol{M}e^{[\boldsymbol{\mathcal{B}}_1]\theta_1}e^{[\boldsymbol{\mathcal{B}}_2]\theta_2}\dots e^{[\boldsymbol{\mathcal{B}}_n]\theta_n}$$

(forward kinematics)

$$[\boldsymbol{\mathcal{V}}_b] = \boldsymbol{T}^{-1}\dot{\boldsymbol{T}} \quad (\boldsymbol{T} = \boldsymbol{T}_{sb})$$

$$\dot{\boldsymbol{T}} = \boldsymbol{M}e^{[\boldsymbol{\mathcal{B}}_1]\theta_1}\dots e^{[\boldsymbol{\mathcal{B}}_{n-1}]\theta_{n-1}}\left(\frac{d}{dt}e^{[\boldsymbol{\mathcal{B}}_n]\theta_n}\right) + \boldsymbol{M}e^{[\boldsymbol{\mathcal{B}}_1]\theta_1}\dots\left(\frac{d}{dt}e^{[\boldsymbol{\mathcal{B}}_{n-1}]\theta_{n-1}}\right)e^{[\boldsymbol{\mathcal{B}}_n]\theta_n} + \cdots$$

$$= \boldsymbol{M}e^{[\boldsymbol{\mathcal{B}}_1]\theta_1}\dots e^{[\boldsymbol{\mathcal{B}}_n]\theta_n}[\boldsymbol{\mathcal{B}}_n]\dot{\theta}_n + \boldsymbol{M}e^{[\boldsymbol{\mathcal{B}}_1]\theta_1}\dots e^{[\boldsymbol{\mathcal{B}}_{n-1}]\theta_{n-1}}[\boldsymbol{\mathcal{B}}_{n-1}]e^{[\boldsymbol{\mathcal{B}}_n]\theta_n}\dot{\theta}_{n-1} + \cdots$$
$$+\boldsymbol{M}e^{[\boldsymbol{\mathcal{B}}_1]\theta_1}[\boldsymbol{\mathcal{B}}_1]e^{[\boldsymbol{\mathcal{B}}_2]\theta_2}\dots e^{[\boldsymbol{\mathcal{B}}_n]\theta_n}\dot{\theta}_1$$

$$\boldsymbol{T}^{-1} = e^{-[\boldsymbol{\mathcal{B}}_n]\theta_n}\dots e^{-[\boldsymbol{\mathcal{B}}_1]\theta_1}\boldsymbol{M}^{-1}$$

$$[\boldsymbol{\mathcal{V}}_b] = [\boldsymbol{\mathcal{B}}_n]\dot{\theta}_n + e^{-[\boldsymbol{\mathcal{B}}_n]\theta_n}[\boldsymbol{\mathcal{B}}_{n-1}]e^{[\boldsymbol{\mathcal{B}}_n]\theta_n}\dot{\theta}_{n-1} + \cdots + e^{-[\boldsymbol{\mathcal{B}}_n]\theta_n}\dots e^{-[\boldsymbol{\mathcal{B}}_2]\theta_2}[\boldsymbol{\mathcal{B}}_1]e^{[\boldsymbol{\mathcal{B}}_2]\theta_2}\dots e^{[\boldsymbol{\mathcal{B}}_n]\theta_n}\dot{\theta}_1$$

$$\boldsymbol{A}^{-1}[\boldsymbol{\mathcal{B}}_i]\boldsymbol{A} = [[\mathrm{Ad}_{\boldsymbol{A}^{-1}}]\boldsymbol{\mathcal{B}}_i] \qquad \boldsymbol{A} \in SE(3)$$

$$\boldsymbol{\mathcal{V}}_b = \underbrace{\boldsymbol{\mathcal{B}}_n}_{\boldsymbol{J}_{bn}}\dot{\theta}_n + \underbrace{\left[\mathrm{Ad}_{e^{-[\boldsymbol{\mathcal{B}}_n]\theta_n}}\right]\boldsymbol{\mathcal{B}}_{n-1}}_{\boldsymbol{J}_{b,n-1}}\dot{\theta}_{n-1} + \cdots + \underbrace{\left[\mathrm{Ad}_{e^{-[\boldsymbol{\mathcal{B}}_n]\theta_n}\cdots e^{-[\boldsymbol{\mathcal{B}}_2]\theta_2}}\right]\boldsymbol{\mathcal{B}}_1}_{\boldsymbol{J}_{b1}}\dot{\theta}_1$$

$$\boldsymbol{\mathcal{V}}_b = \boldsymbol{J}_{b1}(\theta_2,\dots,\theta_n)\dot{\theta}_1 + \cdots + \boldsymbol{J}_{bn-1}(\theta_n)\dot{\theta}_{n-1} + \boldsymbol{J}_{bn}\dot{\theta}_n$$

Body Jacobian (cont.)

$$\mathcal{V}_b = [J_{b_1}(\theta_2, \dots, \theta_n) \quad \cdots \quad J_{b_{n-1}}(\theta_n) \quad J_{b_n}] \begin{bmatrix} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_n \end{bmatrix} = \boldsymbol{J}_b(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$$

The **body Jacobian** $\boldsymbol{J}_b(\boldsymbol{\theta}) \in \mathbb{R}^{6\times n}$ relates the joint rate vector $\dot{\boldsymbol{\theta}} \in \mathbb{R}^n$ to the end-effector (or body) twist $\mathcal{V}_b$. The ith column of $\boldsymbol{J}_b(\boldsymbol{\theta})$ is

$$\boldsymbol{J}_{bi}(\boldsymbol{\theta}) = \begin{bmatrix} \boldsymbol{\omega}_{bi}(\boldsymbol{\theta}) \\ \boldsymbol{v}_{bi}(\boldsymbol{\theta}) \end{bmatrix} = \underbrace{\left[\mathrm{Ad}_{e^{-[\mathcal{B}_n]\theta_n} \dots e^{-[\mathcal{B}_{i+1}]\theta_{i+1}}}\right] \mathcal{B}_i}$$

$$\boldsymbol{J}_{bn} = \mathcal{B}_n$$
$$i = n-1, \dots, 1$$

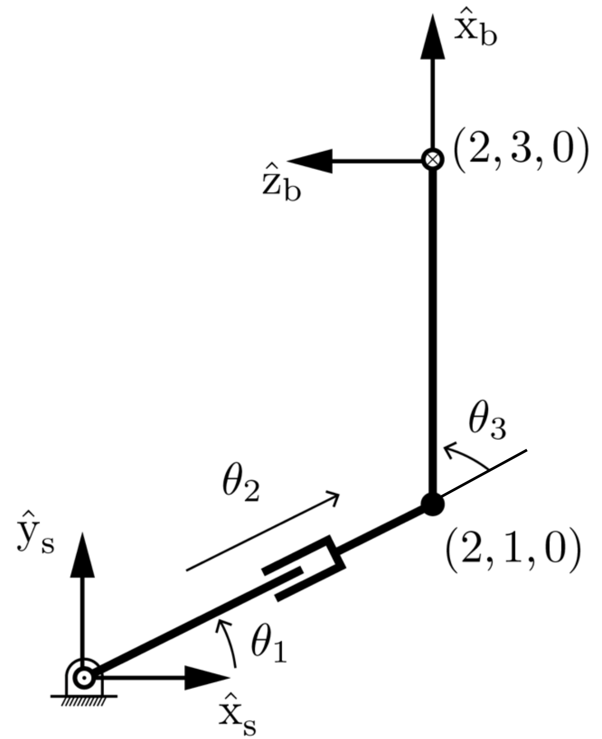
Screw axis describing the ith joint axis (expressed in the end-effector frame $\{b\}$) after the joints $i+1, \dots, n$ move from their zero position to the current values $\theta_n, \dots, \theta_{i+1}$.

Screw axis describing the ith joint axis (expressed in the end-effector frame $\{b\}$) when the robot in its zero/home configuration $\boldsymbol{\theta} = \mathbf{0}$.

Note: The body Jacobian $\boldsymbol{J}_b$ is independent of the choice of the space frame $\{s\}$.
Note: $\boldsymbol{J}_{bi}$ is determined in the same way as the joint screw axis $\mathcal{B}_i$, except that $\boldsymbol{J}_{bi}$ is determined for an arbitrary $\boldsymbol{\theta}$ rather than $\boldsymbol{\theta} = \mathbf{0}$.

Example

Find the space and body Jacobians in the given configuration.



Relationship between Space and Body Jacobian

$$\mathcal{V}_s = J_s(\theta)\dot{\theta}$$
$$\mathcal{V}_b = J_b(\theta)\dot{\theta}$$

$$\mathcal{V}_s = [\mathrm{Ad}_{T_{sb}}]\mathcal{V}_b$$
$$\mathcal{V}_b = [\mathrm{Ad}_{T_{bs}}]\mathcal{V}_s$$
$$[\mathrm{Ad}_{T_1}][\mathrm{Ad}_{T_2}]\mathcal{V} = [\mathrm{Ad}_{T_1T_2}]\mathcal{V}$$

$$\mathcal{V}_s = J_s(\theta)\dot{\theta} \quad \Rightarrow \quad [\mathrm{Ad}_{T_{sb}}]\mathcal{V}_b = J_s(\theta)\dot{\theta} \quad \Rightarrow \quad [\mathrm{Ad}_{T_{bs}}][\mathrm{Ad}_{T_{sb}}]\mathcal{V}_b = [\mathrm{Ad}_{T_{bs}}]J_s(\theta)\dot{\theta}$$

$$\Rightarrow \quad \mathcal{V}_b = [\mathrm{Ad}_{T_{bs}}]J_s(\theta)\dot{\theta} \quad \Rightarrow \quad J_b(\theta)\dot{\theta} = [\mathrm{Ad}_{T_{bs}}]J_s(\theta)\dot{\theta} \quad \overset{\forall\dot{\theta}\neq 0}{\Rightarrow}$$

$$\boxed{J_b(\theta) = [\mathrm{Ad}_{T_{bs}}]J_s(\theta)}$$

Similarly, $\boxed{J_s(\theta) = [\mathrm{Ad}_{T_{sb}}]J_b(\theta)}$

Note: The space and body Jacobians, and the space and body twists, are similarly related by the adjoint map because each column of the space or body Jacobian corresponds to a twist.

Another Form of Geometric Jacobian

Another form of geometric Jacobian is defined as $\begin{bmatrix} \boldsymbol{\omega}_s \\ \dot{\boldsymbol{p}} \end{bmatrix} = \boldsymbol{J}_g(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$

where $\boldsymbol{\omega}_s$ is the angular velocity of EE frame $\{b\}$ expressed in fixed frame $\{s\}$ ans $\dot{\boldsymbol{p}}$ is linear velocity of the origin of EE frame $\{b\}$ expressed in fixed frame $\{s\}$.

Example

Prove that the relationship between the space Jacobian $\boldsymbol{J}_s$ where $\begin{bmatrix} \boldsymbol{\omega}_s \\ \boldsymbol{v}_s \end{bmatrix} = \boldsymbol{J}_s(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$ and geometric Jacobian $\boldsymbol{J}_g$ where $\begin{bmatrix} \boldsymbol{\omega}_s \\ \dot{\boldsymbol{p}} \end{bmatrix} = \boldsymbol{J}_g(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$ is as follows.

$$\boldsymbol{J}_g(\boldsymbol{\theta}) = \begin{bmatrix} \boldsymbol{I}_3 & \boldsymbol{0} \\ -[\boldsymbol{p}] & \boldsymbol{I}_3 \end{bmatrix} \boldsymbol{J}_s(\boldsymbol{\theta})$$

Analytic Jacobian

Alternative Notions of Jacobian

There exist alternative notions of the Jacobian that are based on the representation of the end-effector configuration using a minimum set of coordinates $\boldsymbol{x}_e$ corresponding to a specific robot task space (which is a subspace of $SE(3)$), i.e., the different representations of rotations (e.g., Euler angles $\boldsymbol{\phi}$, unit quaternions $\boldsymbol{q}$, or exponential coordinates $\boldsymbol{r}$), or the different definitions of the end-effector velocities.

- $\boldsymbol{\mathcal{V}}_s = \begin{bmatrix} \boldsymbol{\omega}_s \\ \boldsymbol{v}_s \end{bmatrix} = \boldsymbol{J}_s(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$
- $\boldsymbol{\mathcal{V}}_b = \begin{bmatrix} \boldsymbol{\omega}_b \\ \boldsymbol{v}_b \end{bmatrix} = \boldsymbol{J}_b(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$
- $\begin{bmatrix} \boldsymbol{\omega}_s \\ \dot{\boldsymbol{p}} \end{bmatrix} = \boldsymbol{J}_g(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$

Geometric Jacobian

- $\dot{\boldsymbol{x}}_e = \begin{bmatrix} \dot{\boldsymbol{\phi}} \\ \dot{\boldsymbol{p}} \end{bmatrix} = \dfrac{\partial \begin{bmatrix} \boldsymbol{\phi} \\ \boldsymbol{p} \end{bmatrix}}{\partial \boldsymbol{\theta}} \dot{\boldsymbol{\theta}} = \boldsymbol{J}_{a,\phi}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$ $\qquad \boldsymbol{\phi} = (\alpha, \beta, \gamma)$
- $\dot{\boldsymbol{x}}_e = \begin{bmatrix} \dot{\boldsymbol{q}} \\ \dot{\boldsymbol{p}} \end{bmatrix} = \dfrac{\partial \begin{bmatrix} \boldsymbol{q} \\ \boldsymbol{p} \end{bmatrix}}{\partial \boldsymbol{\theta}} \dot{\boldsymbol{\theta}} = \boldsymbol{J}_{a,q}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$ $\qquad \boldsymbol{q} = (q_0, q_1, q_2, q_3)$ $(\|\boldsymbol{q}\| = 1)$
- $\dot{\boldsymbol{x}}_e = \begin{bmatrix} \dot{\boldsymbol{r}} \\ \dot{\boldsymbol{p}} \end{bmatrix} = \dfrac{\partial \begin{bmatrix} \boldsymbol{r} \\ \boldsymbol{p} \end{bmatrix}}{\partial \boldsymbol{\theta}} \dot{\boldsymbol{\theta}} = \boldsymbol{J}_{a,r}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$ $\qquad \boldsymbol{r} = \hat{\boldsymbol{\omega}}\theta \in \mathbb{R}^3$ $(\|\hat{\boldsymbol{\omega}}\| = 1, \theta \in [0, \pi])$

If the end-effector velocity is represented by the <u>time derivative of the coordinates</u>, the Jacobian is called the **Analytic Jacobian** $\boldsymbol{J}_a$, which is derived by differentiation of the forward kinematics function with respect to the joint variables: $\boldsymbol{x}_e = \mathrm{FK}(\boldsymbol{\theta}) \;\rightarrow\; \dot{\boldsymbol{x}}_e = \boldsymbol{J}_a(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$.

Note: $\boldsymbol{r}$ and $\boldsymbol{q}$ represent the orientation of EE frame $\{b\}$ expressed in fixed frame $\{s\}$ and $\boldsymbol{p}$ represents the position of the origin of $\{b\}$ expressed in $\{s\}$.

Geometric Jacobian vs Analytic Jacobian

- From a physical viewpoint, the meaning of $\boldsymbol{\omega}$ is more intuitive than that of $\dot{\boldsymbol{\phi}}$, $\dot{\boldsymbol{q}}$, and $\dot{\boldsymbol{r}}$. However, while the integral of $\dot{\boldsymbol{\phi}}$, $\dot{\boldsymbol{q}}$, and $\dot{\boldsymbol{r}}$ over time gives $\boldsymbol{\phi}$, $\boldsymbol{q}$, and $\boldsymbol{r}$, the integral of $\boldsymbol{\omega}$ does not admit a clear physical interpretation.

- The geometric Jacobian is used whenever it is necessary to refer to quantities of clear **physical meaning**, while the analytical Jacobian is used whenever it is necessary to refer to differential quantities of variables defined in the task space.

Example

Prove that the relationship between the body Jacobian $\boldsymbol{J}_b$ where $\boldsymbol{\mathcal{V}}_b = \begin{bmatrix} \boldsymbol{\omega}_b \\ \boldsymbol{v}_b \end{bmatrix} = \boldsymbol{J}_b(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$ and an analytic Jacobian $\boldsymbol{J}_{a,r}$ where $\begin{bmatrix} \dot{\boldsymbol{r}} \\ \dot{\boldsymbol{p}} \end{bmatrix} = \boldsymbol{J}_{a,r}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$ is as follows.

$$\boldsymbol{J}_{a,r}(\boldsymbol{\theta}) = \begin{bmatrix} \boldsymbol{A}^{-1}(\boldsymbol{r}) & \boldsymbol{0} \\ \boldsymbol{0} & e^{[\boldsymbol{r}]} \end{bmatrix} \boldsymbol{J}_b(\boldsymbol{\theta})$$

Note: $\boldsymbol{\omega}_b = \boldsymbol{A}(\boldsymbol{r})\dot{\boldsymbol{r}}$ where $\boldsymbol{A}(\boldsymbol{r}) = \boldsymbol{I}_3 - \frac{1-\cos\|\boldsymbol{r}\|}{\|\boldsymbol{r}\|^2}[\boldsymbol{r}] + \frac{\|\boldsymbol{r}\| - \sin\|\boldsymbol{r}\|}{\|\boldsymbol{r}\|^3}[\boldsymbol{r}]^2$

and we assume that the matrix $\boldsymbol{A}(\boldsymbol{r})$ is invertible.

Example

Prove that the relationship between the analytic Jacobian $\boldsymbol{J}_{a,\phi}$, where $\begin{bmatrix} \dot{\boldsymbol{\phi}} \\ \dot{\boldsymbol{p}} \end{bmatrix} = \boldsymbol{J}_{a,\phi}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$ and $\boldsymbol{\phi} = (\varphi, \vartheta, \psi)$ is the Euler angles ZYZ in current frame, and geometric Jacobian $\boldsymbol{J}_g$ where $\begin{bmatrix} \boldsymbol{\omega}_s \\ \dot{\boldsymbol{p}} \end{bmatrix} = \boldsymbol{J}_g(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$ is as follows.

$$\boldsymbol{J}_g(\boldsymbol{\theta}) = \begin{bmatrix} \boldsymbol{A}(\boldsymbol{\phi}) & \mathbf{0}_3 \\ \mathbf{0}_3 & \boldsymbol{I}_3 \end{bmatrix} \boldsymbol{J}_{a,\phi}(\boldsymbol{\theta})$$

General Form of Velocity Kinematics

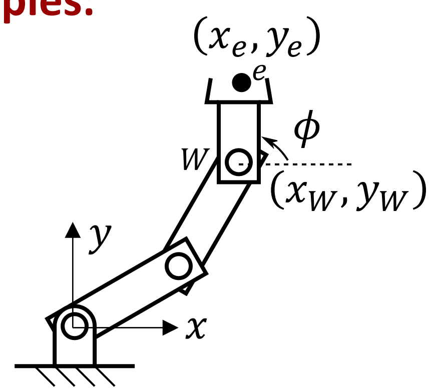
In general, depending on the dimension of task space r ($r \leq 6$), the differential kinematics equation can be represented as

$$\boldsymbol{\mathcal{V}} = \boldsymbol{J}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$$

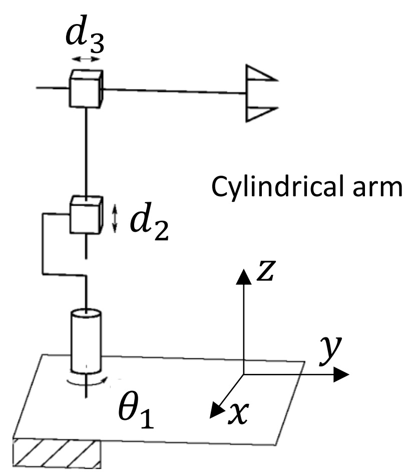
where now $\boldsymbol{\mathcal{V}} \in \mathbb{R}^r$ (e.g., $\boldsymbol{\mathcal{V}}_b$, $\boldsymbol{\mathcal{V}}_s$, $(\boldsymbol{\omega}_s, \dot{\boldsymbol{p}})$, or $\dot{\boldsymbol{x}}_e$) is end-effector velocity for the specific task, $\dot{\boldsymbol{\theta}} \in \mathbb{R}^n$, and $\boldsymbol{J}(\boldsymbol{\theta}) \in \mathbb{R}^{r\times n}$ is the corresponding Jacobian matrix that can be extracted from a $6 \times n$ geometric or analytic Jacobian (by removing null and irrelevant rows).

Examples:

(1)

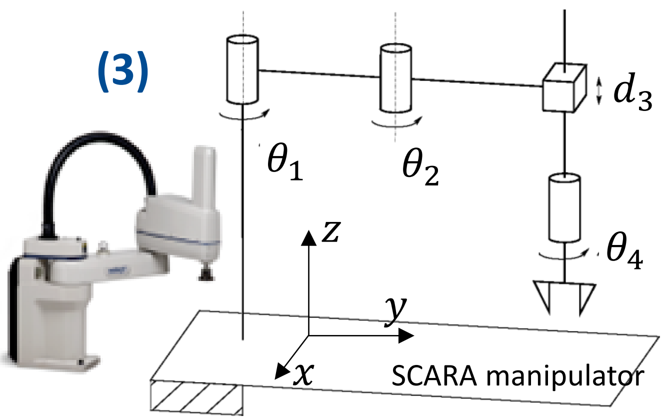


(2)



$\boldsymbol{J} \in \mathbb{R}^{3\times 3}$

(3)



$\boldsymbol{J} \in \mathbb{R}^{4\times 4}$

- For the 3R planar robot (1) with $\boldsymbol{x}_e = (x_W, y_W, \phi)$, $\boldsymbol{J} \in \mathbb{R}^{3\times 3}$ and with $\boldsymbol{x}_e = (x_e, y_e)$, $\boldsymbol{J} \in \mathbb{R}^{2\times 3}$.
- (2) and (3) are inherently impossible to rotate about axes x and y.

Singularity Analysis

Kinematic Singularity

as we are interested in the velocity of $\{b\}$ rather than $\{s\}$.

The configurations at which the robot's end-effector loses the ability to move instantaneously in one or more directions is called a **Kinematic Singularity**. In these directions, the robot can resist arbitrary wrenches.

- In singular configuration $\boldsymbol{\theta}^*$, $\boldsymbol{J}(\boldsymbol{\theta}^*) \in \mathbb{R}^{r\times n}$ is rank-deficient, i.e., $\operatorname{rank}\big(\boldsymbol{J}(\boldsymbol{\theta}^*)\big) < \min(r,n)$.

  To check rank-deficiency, use the Jacobian that maps $\dot{\boldsymbol{\theta}}$ to the non-zero and independent velocities of the EE frame $\{b\}$ (i.e., after removing null and irrelevant rows of $\boldsymbol{J}(\boldsymbol{\theta}) \in \mathbb{R}^{r\times n}$ where $\boldsymbol{J}$ can be $\boldsymbol{J}_b, \boldsymbol{J}_g$, or $\boldsymbol{J}_a$ as we are interested in the velocity of $\{b\}$ rather than $\{s\}$).

- The kinematic singularities are typical of a mechanical structure and independent of the choice of the frames (e.g., fixed frame $\{s\}$ and end-effector frame $\{b\}$).

- In the neighborhood of a singularity, small velocities $\boldsymbol{\mathcal{V}}$ in the task space may cause large velocities $\dot{\boldsymbol{\theta}}$ in the joint space.

- Since $[\mathrm{Ad}_{\boldsymbol{T}}]$ is always invertible and $\boldsymbol{J}_s = [\mathrm{Ad}_{\boldsymbol{T}_{sb}}]\boldsymbol{J}_b$, $\boldsymbol{J}_b$ and $\boldsymbol{J}_s$ always have the same rank.

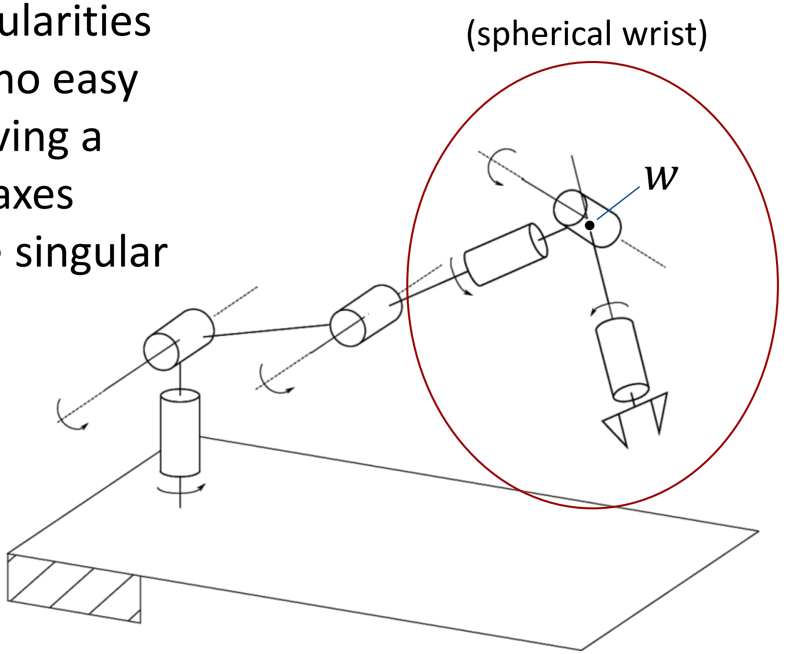
Kinematic Singularity

Singularities can be classified into:

- **Boundary Singularities**: They occur when the manipulator is either outstretched or retracted (it is easy to avoid).
- **Internal Singularities**: They occur anywhere inside the reachable workspace (it is hard to avoid).

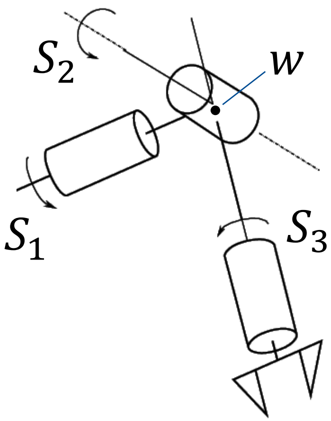
Singularity Decoupling: Computation of internal singularities via the Jacobian determinant may be tedious and of no easy solution for complex structures. For manipulators having a spherical wrist (i.e., three consecutive revolute joint axes intersect at a common point w), we can compute the singular configurations in two steps:

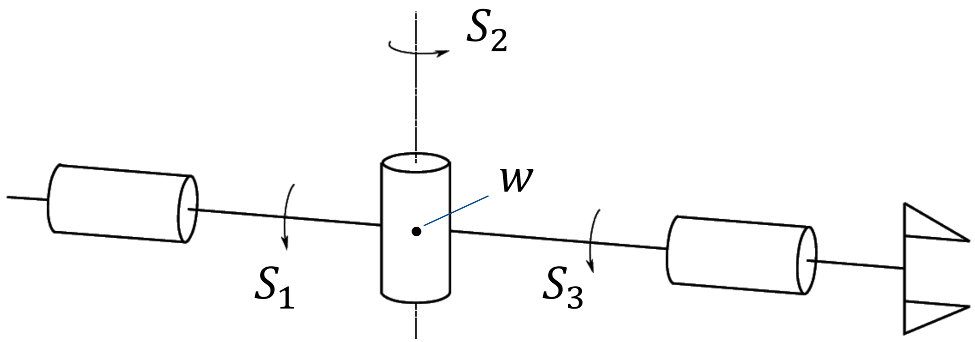
- Computation of **wrist singularities** resulting from the motion of the spherical wrist.
- Computation of **arm singularities** resulting from the motion of the first 3 or more links.



An Example of Wrist Singularity

The singularity occurs when S_1 and S_3 are aligned.



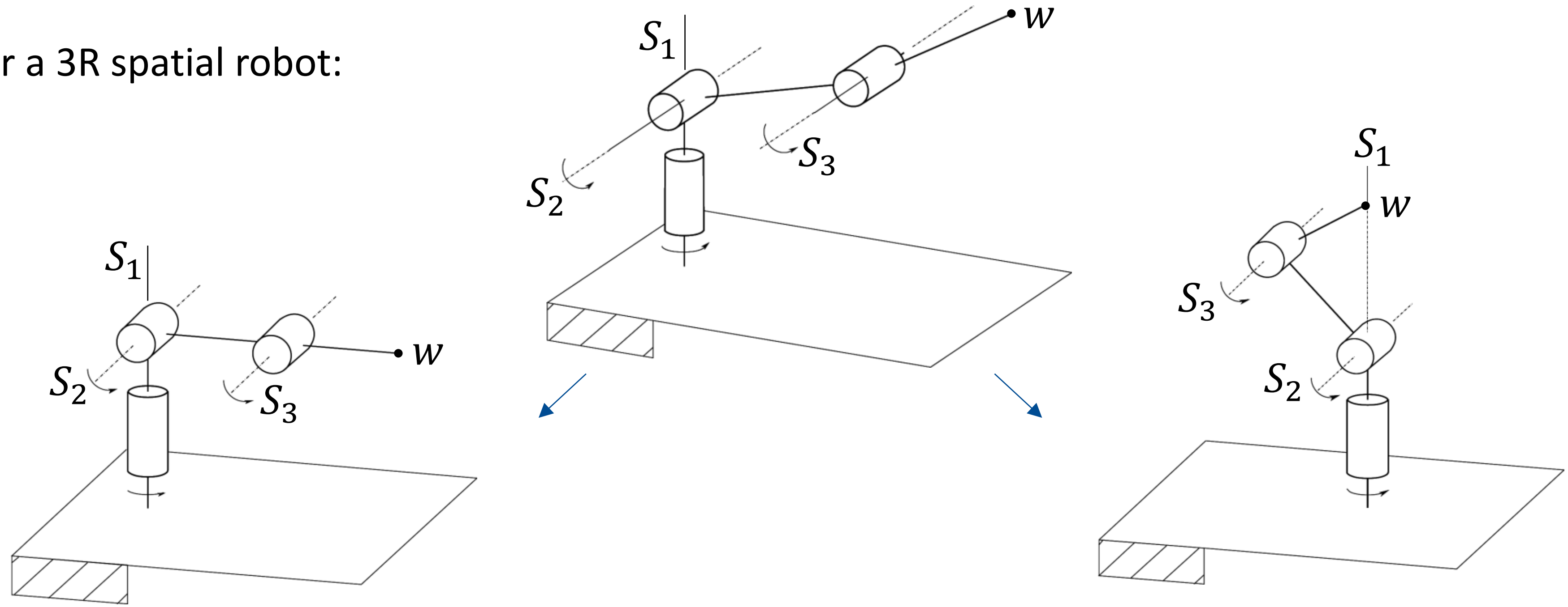


- In this configuration, the wrist cannot rotate about the axis orthogonal to S_1 and S_2.
- Rotations of equal magnitude about opposite directions on S_1 and S_3 do not produce any end-effector rotation.

Note: Wrist singularity is naturally described in the joint space and can be encountered anywhere inside the manipulator reachable workspace.

Examples of Arm Singularities

For a 3R spatial robot:



Elbow Singularity: when the elbow is outstretched or retracted.

Shoulder Singularity: when the wrist point (w) lies on axis S_1 (the whole axis S_1 describes a continuum of singular configurations).

Note: Arm singularity is well identified in the task space, and thus, they can be suitably avoided in the end-effector trajectory planning stage.

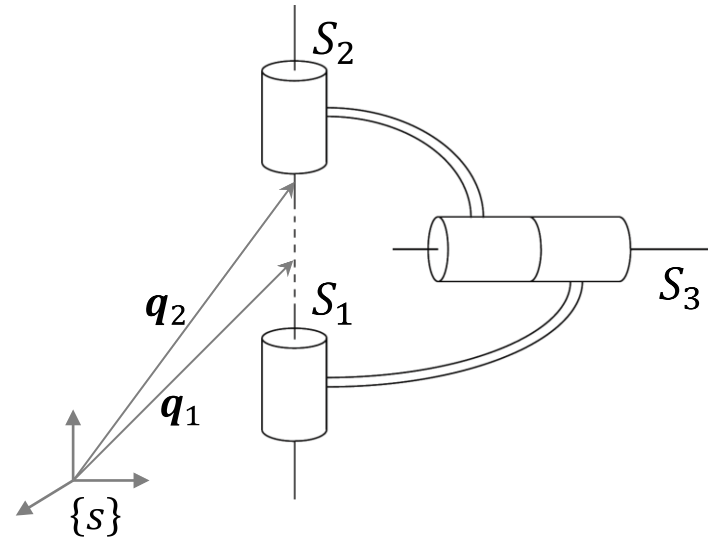
Examples of Common Singular Configurations ($n \geq 3$)

Case I: Two <u>Collinear</u> Revolute Joint Axes

$$J_{s1}(\theta) = \begin{bmatrix} \omega_{s1} \\ -\omega_{s1} \times q_1 \end{bmatrix} \qquad J_{s2}(\theta) = \begin{bmatrix} \omega_{s2} \\ -\omega_{s2} \times q_2 \end{bmatrix}$$

$$\left.\begin{aligned} \omega_{s1} &= \omega_{s2} \\ \omega_{s1} \times q_1 &= \omega_{s1} \times q_1 \end{aligned}\right\} J_{s1} = J_{s2}$$

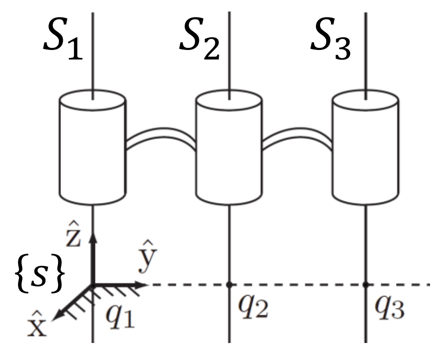
The set $\{J_{s1}, J_{s2}, \ldots\}$ cannot be linearly independent.



Case II: Three <u>Coplanar and Parallel</u> Revolute Joint Axes

$$J_s(\theta) = \begin{bmatrix} \omega_{s1} & \omega_{s1} & \omega_{s1} & \cdots \\ 0 & \underbrace{-\omega_{s1} \times q_2}_{u} & \underbrace{-\omega_{s1} \times q_3}_{\alpha u} & \cdots \end{bmatrix}$$

The set $\{J_{s1}, J_{s2}, J_{s3}, \ldots\}$ cannot be linearly independent.

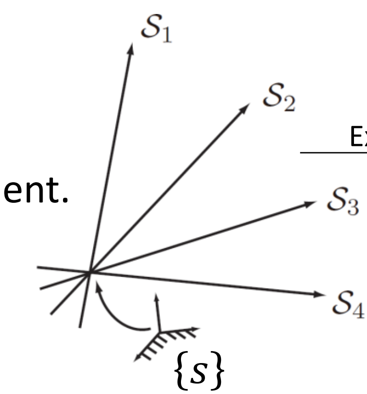


Examples of Common Singular Configurations ($n \geq 4$)

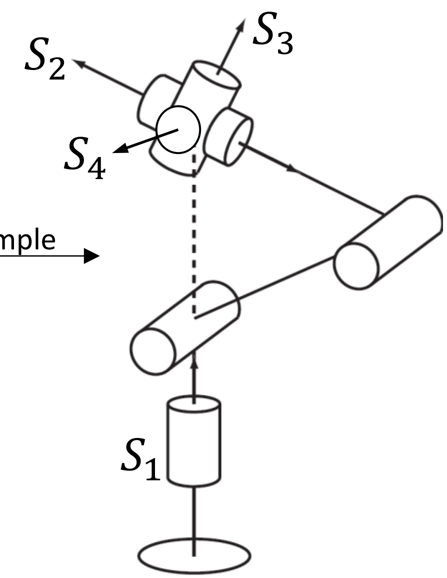
Case III: Four Revolute Joint Axes Intersecting at a Common Point

$$J_s(\boldsymbol{\theta}) = \begin{bmatrix} \boldsymbol{\omega}_{s1} & \boldsymbol{\omega}_{s2} & \boldsymbol{\omega}_{s3} & \boldsymbol{\omega}_{s4} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \end{bmatrix}$$

The set $\{J_{s1}, J_{s2}, J_{s3}, J_{s4}, \ldots\}$ cannot be linearly independent.



Example



Case IV: Four Coplanar Revolute Joints

$$\boldsymbol{\omega}_{si} = \begin{bmatrix} \omega_{six} \\ \omega_{siy} \\ 0 \end{bmatrix} \qquad \boldsymbol{q}_i = \begin{bmatrix} q_{ix} \\ q_{iy} \\ 0 \end{bmatrix} \qquad -\boldsymbol{\omega}_{si} \times \boldsymbol{q}_i = \begin{bmatrix} 0 \\ 0 \\ \omega_{siy} q_{ix} - \omega_{six} q_{iy} \end{bmatrix}$$

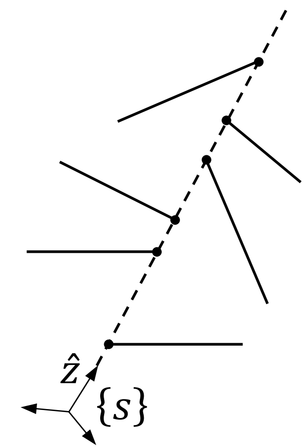
$$J_s(\boldsymbol{\theta}) = \begin{bmatrix} \omega_{s1x} & \omega_{s2x} & \omega_{s3x} & \omega_{s4x} \\ \omega_{s1y} & \omega_{s2y} & \omega_{s3y} & \omega_{s4y} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \omega_{s1y} q_{1x} - \omega_{s1x} q_{1y} & \omega_{s2y} q_{2x} - \omega_{s2x} q_{2y} & \omega_{s3y} q_{3x} - \omega_{s3x} q_{3y} & \omega_{s4y} q_{4x} - \omega_{s4x} q_{4y} \end{bmatrix}$$

The set $\{J_{s1}, J_{s2}, J_{s3}, J_{s4}, \ldots\}$ cannot be linearly independent.

Examples of Common Singular Configurations ($n \geq 6$)

Case V: Six Revolute Joints Intersecting a Common Line

$$-\boldsymbol{\omega}_{si} \times \boldsymbol{q}_i = (\omega_{siy} q_{iz}, -\omega_{six} q_{iz}, 0)$$

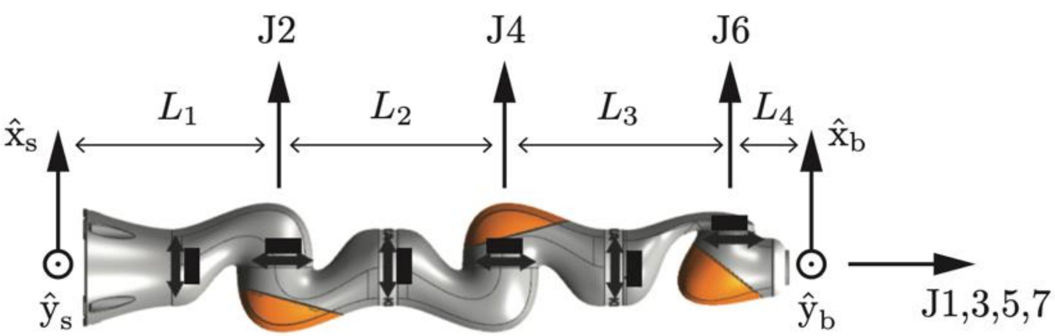


$$\boldsymbol{J}_s(\boldsymbol{\theta}) = \begin{bmatrix} \omega_{s1x} & \omega_{s2x} & \omega_{s3x} & \omega_{s4x} & \omega_{s5x} & \omega_{s6x} \\ \omega_{s1y} & \omega_{s2y} & \omega_{s3y} & \omega_{s4y} & \omega_{s5y} & \omega_{s6y} \\ \omega_{s1z} & \omega_{s2z} & \omega_{s3z} & \omega_{s4z} & \omega_{s5z} & \omega_{s6z} \\ \omega_{s1y} q_{1z} & \omega_{s2y} q_{2z} & \omega_{s3y} q_{3z} & \omega_{s4y} q_{4z} & \omega_{s5y} q_{5z} & \omega_{s6y} q_{6z} \\ -\omega_{s1x} q_{1z} & -\omega_{s2x} q_{2z} & -\omega_{s3x} q_{3z} & -\omega_{s4x} q_{4z} & -\omega_{s5x} q_{5z} & -\omega_{s6x} q_{6z} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The set $\{\boldsymbol{J}_{s1}, \boldsymbol{J}_{s2}, \boldsymbol{J}_{s3}, \boldsymbol{J}_{s4}, \boldsymbol{J}_{s5}, \boldsymbol{J}_{s6}, \dots\}$ cannot be linearly independent.

Example

For the KUKA LBR iiwa 7R robot shown in its zero/home configuration, for a general task of manipulating a rigid body, what is the dimension of the Jacobian matrix? What is the rank of the Jacobian at this configuration?



Inverse Velocity Kinematics and Redundancy Analysis

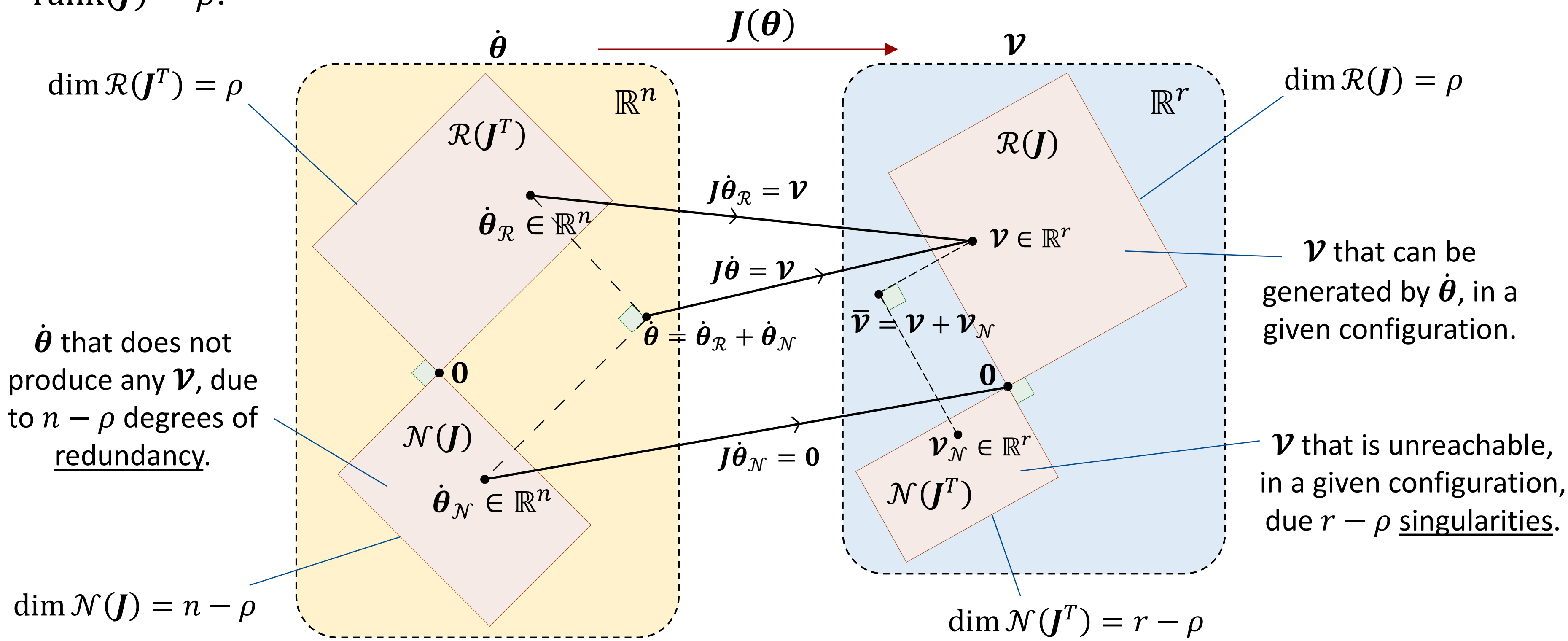
Analysis of Velocity Kinematics and Kinematic Redundancy

Assume a n-DOF robot that is not at a singular configuration and $\dim(\text{T–Space}) = r$,

- If $n < r$, then arbitrary twists $\boldsymbol{\mathcal{V}}$ cannot be achieved (the robot does not have enough joints).
- If $n = r$, then any arbitrary twists $\boldsymbol{\mathcal{V}}$ can be achieved (the robot have enough joints).
- If $n > r$, then not only any arbitrary twists $\boldsymbol{\mathcal{V}}$ can be achieved but also the remaining $n - r$ degrees of freedoms (redundant DOFs) can generates **internal motions** at the joints of the robot that are not evident in the motion of the end-effector.

Analysis of Velocity Kinematics and Kinematic Redundancy

Consider the velocity kinematics equation $\mathcal{V} = J(\theta)\dot{\theta}$ where $J(\theta) \in \mathbb{R}^{r\times n}$, $\dot{\theta} \in \mathbb{R}^n$, $\mathcal{V} \in \mathbb{R}^r$, and $\mathrm{rank}(J) = \rho$.



- If the Jacobian is full-rank (robot is not at a singular configuration) and $n > r$ (robot is redundant): $\dim \mathcal{R}(J) = r$, $\dim \mathcal{N}(J) = n - r$, and $\dim \mathcal{N}(J^T) = 0$.

Inverse Velocity Kinematics

Given a desired EE twist $\boldsymbol{\mathcal{V}} \in \mathbb{R}^r$, what joint velocities $\dot{\boldsymbol{\theta}} \in \mathbb{R}^n$ are needed?

- If $\boldsymbol{J}(\boldsymbol{\theta}) \in \mathbb{R}^{r\times n}$ is square ($n = r$) and full rank $\mathrm{rank}(\boldsymbol{J}) = r$, (i.e., not at a singular configuration), then $\boldsymbol{J}(\boldsymbol{\theta})$ is invertible and there is a unique solution as $\dot{\boldsymbol{\theta}} = \boldsymbol{J}(\boldsymbol{\theta})^{-1}\boldsymbol{\mathcal{V}}$.
- If $\boldsymbol{J}(\boldsymbol{\theta}) \in \mathbb{R}^{r\times n}$ is not square and $n > r$ (i.e., robot is redundant, $\boldsymbol{J}$ is a fat matrix, and $\mathcal{N}(\boldsymbol{J}) \neq \emptyset$), and also $\boldsymbol{J}$ is full (row) rank, $\mathrm{rank}(\boldsymbol{J}) = r$ (i.e., robot is not at a singular configuration and $\mathcal{N}(\boldsymbol{J}^T) = \emptyset$), then infinite exact solutions $\dot{\boldsymbol{\theta}}$ exist to $\boldsymbol{\mathcal{V}} = \boldsymbol{J}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$ as

$$\dot{\boldsymbol{\theta}} = \dot{\boldsymbol{\theta}}^* + \boldsymbol{P}\dot{\boldsymbol{\theta}}_0 = \boldsymbol{J}(\boldsymbol{\theta})^+\boldsymbol{\mathcal{V}} + \left(\boldsymbol{I}_n - \boldsymbol{J}(\boldsymbol{\theta})^+\boldsymbol{J}(\boldsymbol{\theta})\right)\dot{\boldsymbol{\theta}}_0 \qquad \forall \dot{\boldsymbol{\theta}}_0 \in \mathbb{R}^n$$

where $\boldsymbol{J}^+ = \boldsymbol{J}^T(\boldsymbol{J}\boldsymbol{J}^T)^{-1}$ is the right pseudo-inverse as $\boldsymbol{J}\boldsymbol{J}^+ = \boldsymbol{I}_r$.

❖ The solution $\dot{\boldsymbol{\theta}}^* = \boldsymbol{J}^+\boldsymbol{\mathcal{V}}$ locally minimizes the norm of joint velocities $\dot{\boldsymbol{\theta}}$.

❖ The matrix $\boldsymbol{P} = \boldsymbol{I}_n - \boldsymbol{J}^+\boldsymbol{J}$ projects of $\dot{\boldsymbol{\theta}}_0$ in $\mathcal{N}(\boldsymbol{J})$, so as not to violate the constraint $\boldsymbol{\mathcal{V}} = \boldsymbol{J}\dot{\boldsymbol{\theta}}$.

Remarks

- $\dot{\boldsymbol{\theta}}_0$ is a vector of arbitrary joint velocities that can generate **internal motions** and can be specified to satisfy an additional constraint to the problem due to the presence of redundant DOFs. The additional constraint has <u>secondary priority</u> with respect to the primary kinematic constraint $\boldsymbol{\mathcal{V}} = \boldsymbol{J}\dot{\boldsymbol{\theta}}$.

- The use of the pseudoinverse $\boldsymbol{J}^{+} = \boldsymbol{J}^{T}(\boldsymbol{J}\boldsymbol{J}^{T})^{-1}$ implicitly weights the cost of each joint velocity identically. We could instead give the joint velocities different weights; for example, the velocity at the first joint, which moves a lot of the robot's mass, could be weighted more heavily than the velocity at the last joint, which moves little of the robot's mass. Therefore, we can use the weighted right pseudo-inverse as

$$\boldsymbol{J}^{+} = \boldsymbol{W}_r^{-1}\boldsymbol{J}^{T}(\boldsymbol{J}\boldsymbol{W}_r^{-1}\boldsymbol{J}^{T})^{-1}$$

$\boldsymbol{W}_r \in \mathbb{R}^{n\times n}$ is a positive definite matrix.

For example, by using $\boldsymbol{W}_r = \boldsymbol{M}(\boldsymbol{\theta})$ where $\boldsymbol{M}(\boldsymbol{\theta})$ is the mass matrix of the robot, we can find the $\dot{\boldsymbol{\theta}}$ that minimizes the kinetic energy, while also satisfying $\boldsymbol{\mathcal{V}} = \boldsymbol{J}\dot{\boldsymbol{\theta}}$.

Exploiting Redundant DOFs
(Redundancy Resolution)

A typical choice of $\dot{\boldsymbol{\theta}}_0$ for advantageously exploiting redundant DOFs is

$$\dot{\boldsymbol{\theta}}_0 = k_0 \nabla_{\boldsymbol{\theta}} w(\boldsymbol{\theta}) = k_0 \left(\frac{\partial w(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right)^T \quad \text{where } k_0 \in \mathbb{R}_+$$

which is in the direction of the gradient of a (secondary) objective function $w(\boldsymbol{\theta})$ at a given $\boldsymbol{\theta}$ (i.e., in the direction at which function $w(\boldsymbol{\theta})$ increases the fastest). Thus, the solution $\dot{\boldsymbol{\theta}} = \dot{\boldsymbol{\theta}}^* + \boldsymbol{P}\dot{\boldsymbol{\theta}}_0$ attempts to maximize $w(\boldsymbol{\theta})$ locally compatible to the primary objective $\boldsymbol{\mathcal{V}} = \boldsymbol{J}\dot{\boldsymbol{\theta}}$ (kinematic constraint).

❖ Three typical (secondary) objective functions $w(\boldsymbol{\theta})$:

1) **Manipulability measure**: $\quad w(\boldsymbol{\theta}) = \sqrt{\det(\boldsymbol{J}(\boldsymbol{\theta})\boldsymbol{J}(\boldsymbol{\theta})^T)}$

By maximizing $w(\boldsymbol{\theta})$, redundancy is exploited to move away from singularities. Note that $w(\boldsymbol{\theta})$ vanishes at a singular configuration.

Exploiting Redundant DOFs
(Redundancy Resolution)

2) **Distance from mechanical joint limits**:

$$w(\boldsymbol{\theta}) = -\frac{1}{2n}\sum_{i=1}^{n}\left(\frac{\theta_i - \bar{\theta}_i}{\theta_{iM} - \theta_{im}}\right)^2$$

n: number of joints,
θ_{iM} (θ_{im}): maximum (minimum) joint limit,
$\bar{\theta}_i$: middle value of the joint range.

By maximizing $w(\boldsymbol{\theta})$, redundancy is exploited to keep the joint variables $\boldsymbol{\theta}$ as close as possible to the center of their ranges.

3) **Distance from an obstacle**:

$$w(\boldsymbol{\theta}) = \min_{\boldsymbol{p}_{\mathcal{B}},\boldsymbol{o}} \|\boldsymbol{p}_{\mathcal{B}}(\boldsymbol{\theta}) - \boldsymbol{o}\|$$

$\boldsymbol{o}$: position vector of a suitable point on the obstacle,
$\boldsymbol{p}_{\mathcal{B}}$: position vector of a generic point along the body $\mathcal{B}$ of the robot.

By maximizing $w(\boldsymbol{\theta})$, redundancy is exploited to avoid collision of the manipulator's body with an obstacle.

Inverse Velocity Kinematics
Kinematic Singularities

- When $\boldsymbol{J}(\boldsymbol{\theta}) \in \mathbb{R}^{r\times n}$ (square or non-square) is rank deficient (i.e., mathematically, $\text{rank}(\boldsymbol{J}) < \min(r, n)$, $\mathcal{N}(\boldsymbol{J}) \neq \emptyset$, and $\mathcal{N}(\boldsymbol{J}^{\mathrm{T}}) \neq \emptyset$, and physically, a redundant or non-redundant robot at a singular configuration),
  - If $\boldsymbol{\mathcal{V}} \in \mathcal{R}(\boldsymbol{J})$, there will be infinite exact solutions in the form

$$\dot{\boldsymbol{\theta}} = \boldsymbol{J}(\boldsymbol{\theta})^{+}\boldsymbol{\mathcal{V}} + \left(\boldsymbol{I}_n - \boldsymbol{J}(\boldsymbol{\theta})^{+}\boldsymbol{J}(\boldsymbol{\theta})\right)\dot{\boldsymbol{\theta}}_0 \qquad \forall \dot{\boldsymbol{\theta}}_0 \in \mathbb{R}^n$$

and $\boldsymbol{J}(\boldsymbol{\theta})^{+}$ is pseudo-inverse which can be computed using the **Singular Value Decomposition (SVD)** or approximately using **Damped Least Squares (DLS)**.

This means that the assigned path is physically executable by the manipulator, even though it is at a singular configuration.

  - If $\boldsymbol{\mathcal{V}} \notin \mathcal{R}(\boldsymbol{J})$, there will be not exact solutions.

This means that the end-effector path cannot be executed by the manipulator at the given posture.

Inverse Velocity Kinematics

Kinematic Singularities

- In the neighborhood of a singularity, small velocities $\mathcal{V}$ in the task space may cause large velocities $\dot{\boldsymbol{\theta}}$ in the joint space.

- The pseudo-inverse $\boldsymbol{J}(\boldsymbol{\theta})^{+}$ computed using the **Damped Least Squares (DLS)** is more stable and computationally less expensive.

$$\boldsymbol{J}^{+}(\boldsymbol{\theta}) = \boldsymbol{J}(\boldsymbol{\theta})^{T}(\boldsymbol{J}(\boldsymbol{\theta})\boldsymbol{J}(\boldsymbol{\theta})^{T} + \lambda \boldsymbol{I}_{r})^{-1}$$

This solution is derived from this optimization problem:

$$\min_{\dot{\boldsymbol{\theta}}} \quad \left(\mathcal{V} - \boldsymbol{J}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}\right)^{T}\left(\mathcal{V} - \boldsymbol{J}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}\right) + \frac{1}{2}\lambda\dot{\boldsymbol{\theta}}^{T}\dot{\boldsymbol{\theta}}$$

The damping factor $\lambda \in \mathbb{R}_{+}$ establishes the relative weight between the kinematic constraint $\mathcal{V} = \boldsymbol{J}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$ and the minimum norm joint velocity requirement. In the neighborhood of a singularity, λ is to be chosen large enough so as to render differential kinematics inversion well conditioned, whereas far from singularities, λ can be chosen small (even $\lambda = 0$) so as to guarantee accurate differential kinematics inversion. There exist techniques for selecting optimal values for λ.

Statics of Open Chains

Statics of Open Chains

The goal of statics is to determine the relationship between wrench $\boldsymbol{\mathcal{F}}$ applied to the end-effector and joint torques $\boldsymbol{\tau} \in \mathbb{R}^n$ applied to the joints (forces for prismatic joints, torques for revolute joints) with the manipulator at an equilibrium configuration.

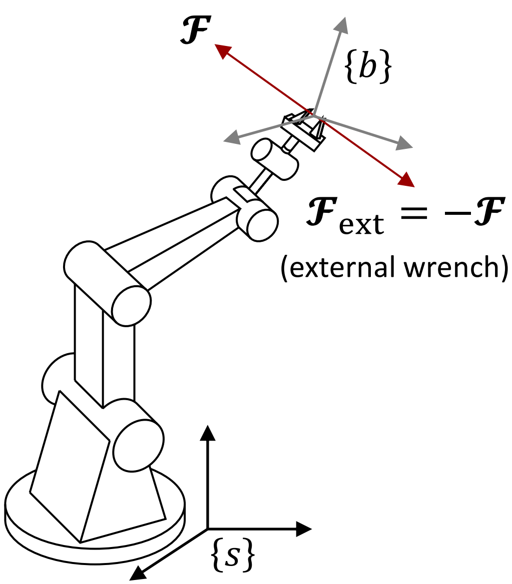
Principle of conservation of power:
power generated at the joints =
(power measured at the end-effector) + (power to move the robot)

At static equilibrium, no power is being used to move the robot, thus:

$$\boldsymbol{\tau}^T \dot{\boldsymbol{\theta}} = \boldsymbol{\mathcal{F}}^T \boldsymbol{\mathcal{V}} \qquad \dot{\boldsymbol{\theta}} \to \mathbf{0}$$

$$\boldsymbol{\mathcal{V}} = \boldsymbol{J}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$$

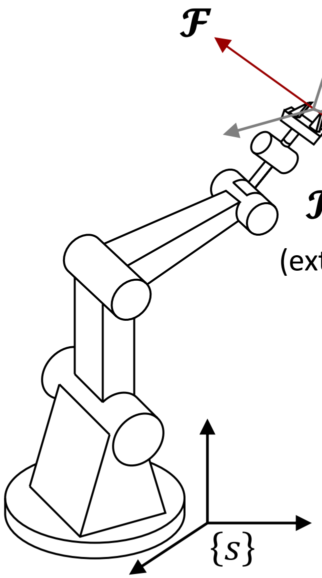
$$\boldsymbol{\tau} = \boldsymbol{J}^T(\boldsymbol{\theta})\boldsymbol{\mathcal{F}}$$



where $\boldsymbol{\mathcal{F}} \in \mathbb{R}^r$ is the wrench generated by the robot, $\boldsymbol{J} \in \mathbb{R}^{r \times n}$ is the corresponding geometric or analytic Jacobian matrix, and $\dim(\text{T–Space}) = r \ (r \leq 6)$.

Statics and Kinematic Redundancy

Note: If an external wrench $\mathcal{F}_{\text{ext}}$ is applied to the end-effector when the robot is at equilibrium, $\boldsymbol{\tau} = \boldsymbol{J}^T \boldsymbol{\mathcal{F}}$ calculates the joint torques $\boldsymbol{\tau}$ needed to generate the opposing wrench $\boldsymbol{\mathcal{F}}$, keeping the robot at equilibrium.



Note: If the robot has to support itself against gravity to maintain static equilibrium, the torques $\boldsymbol{\tau}_{\mathcal{F}} = \boldsymbol{J}^T \boldsymbol{\mathcal{F}}$ must be added to the torques $\boldsymbol{\tau}_g$ that compensate gravity:

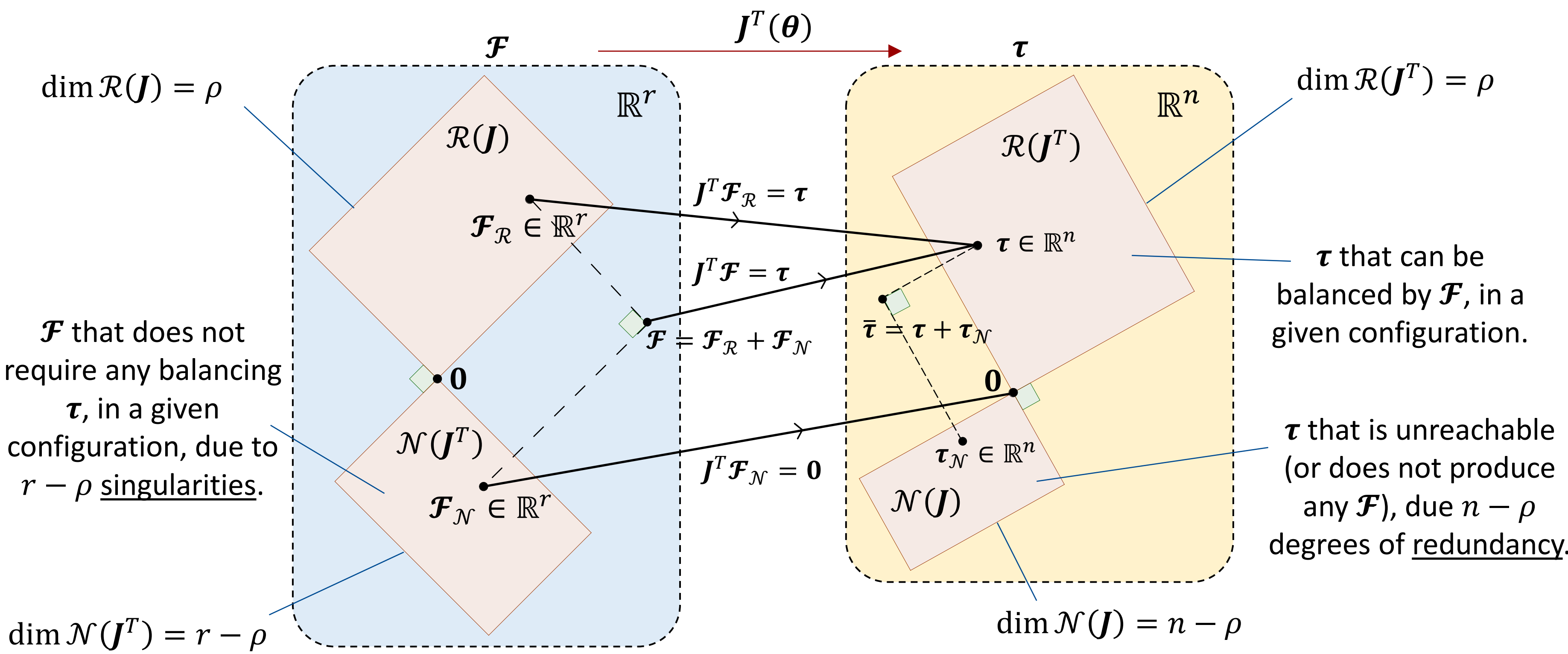
$$\boldsymbol{\tau}_{\text{joint}} = \boldsymbol{J}^T \boldsymbol{\mathcal{F}} + \boldsymbol{\tau}_g$$

Assume a n-DOF robot that is not at a singular configuration and $\dim(\text{T–Space}) = r$,

- If $n = r$, the embedding the end-effector in concrete will immobilize the robot.
- If $n > r$, then the robot is redundant, and even if the end-effector is embedded in concrete, the joint torques may cause internal motions of the links. The static equilibrium assumption is no longer satisfied, and we need to include dynamics to know what will happen to the robot.

Kineto-Statics Duality

Consider static equation $\boldsymbol{\tau} = \boldsymbol{J}^T(\boldsymbol{\theta})\boldsymbol{\mathcal{F}}$ where $\boldsymbol{J}(\boldsymbol{\theta}) \in \mathbb{R}^{r\times n}$, $\boldsymbol{\tau} \in \mathbb{R}^n$, $\boldsymbol{\mathcal{F}} \in \mathbb{R}^r$, and rank$(\boldsymbol{J}) = \rho$.



- If the Jacobian is full-rank (robot is not at a singular configuration) and $n > r$ (robot is redundant): $\dim \mathcal{R}(\boldsymbol{J}) = r$, $\dim \mathcal{N}(\boldsymbol{J}) = n - r$, and $\dim \mathcal{N}(\boldsymbol{J}^T) = 0$.

Manipulability

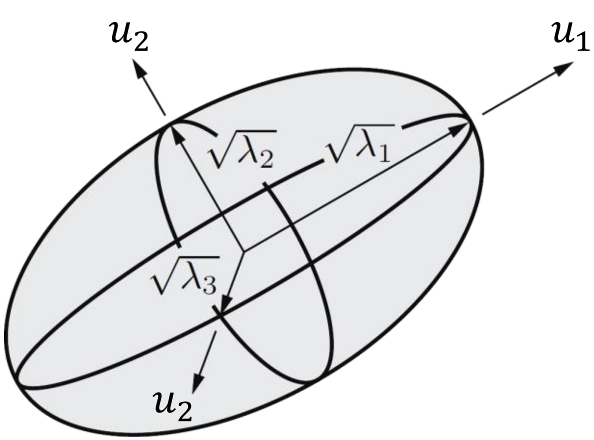
Preliminary: Ellipsoid Representation

For any symmetric positive-definite $\boldsymbol{A} \in \mathbb{R}^{m \times m}$, the set of vectors $\boldsymbol{x} \in \mathbb{R}^{m}$ satisfying $\boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x} = 1$ defines an ellipsoid (function of $\boldsymbol{x}$) in the m-dimensional space.

Assume that $\boldsymbol{u}_i \in \mathbb{R}^m$ are eigenvectors and $\lambda_i \in \mathbb{R}$ are eigenvalues of $\boldsymbol{A}^{-1}$ $(i = 1, \dots, m)$.

Therefore, for the ellipsoid,

- Directions of the principal axes are $\boldsymbol{u}_i$,
- Lengths of the principal semi-axes are $\sqrt{\lambda_i}$,
- Volume is proportional to $\sqrt{\lambda_1 \lambda_2 \cdots \lambda_m} = \sqrt{\det(\boldsymbol{A}^{-1})}$.



Velocity Manipulability Ellipsoid

The **velocity manipulability ellipsoid** corresponds to the end-effector velocities $\boldsymbol{\mathcal{V}} \in \mathbb{R}^r$ for joint rates $\dot{\boldsymbol{\theta}} \in \mathbb{R}^n$ satisfying $\|\dot{\boldsymbol{\theta}}\| = \dot{\boldsymbol{\theta}}^T\dot{\boldsymbol{\theta}} = 1$ (the points on the surface of a sphere).

At a nonsingular configuration:

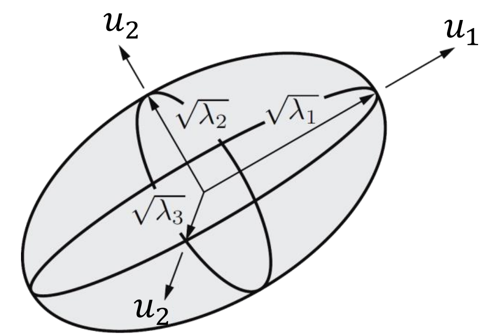
$$\boldsymbol{\mathcal{V}} = \boldsymbol{J}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} \qquad \boldsymbol{\mathcal{V}} \in \mathbb{R}^r, \dot{\boldsymbol{\theta}} \in \mathbb{R}^n, \boldsymbol{J} \in \mathbb{R}^{r\times n}$$

$$\boldsymbol{J}^+ = \boldsymbol{J}^T\left(\boldsymbol{J}\boldsymbol{J}^T\right)^{-1}$$

$$n \geq r$$

$$\begin{aligned} 1 &= \dot{\boldsymbol{\theta}}^T\dot{\boldsymbol{\theta}} \\ &= (\boldsymbol{J}^+\boldsymbol{\mathcal{V}})^T(\boldsymbol{J}^+\boldsymbol{\mathcal{V}}) \\ &= \boldsymbol{\mathcal{V}}^T\boldsymbol{J}^{+T}\boldsymbol{J}^+\boldsymbol{\mathcal{V}} \\ &= \boldsymbol{\mathcal{V}}^T(\boldsymbol{J}\boldsymbol{J}^T)^{-1}\boldsymbol{\mathcal{V}} \end{aligned}$$

(the points on the surface of an ellipsoid)



$\boldsymbol{J}\boldsymbol{J}^T \in \mathbb{R}^{r\times r}$ is square, symmetric, and positive definite, as is $(\boldsymbol{J}\boldsymbol{J}^T)^{-1}$.

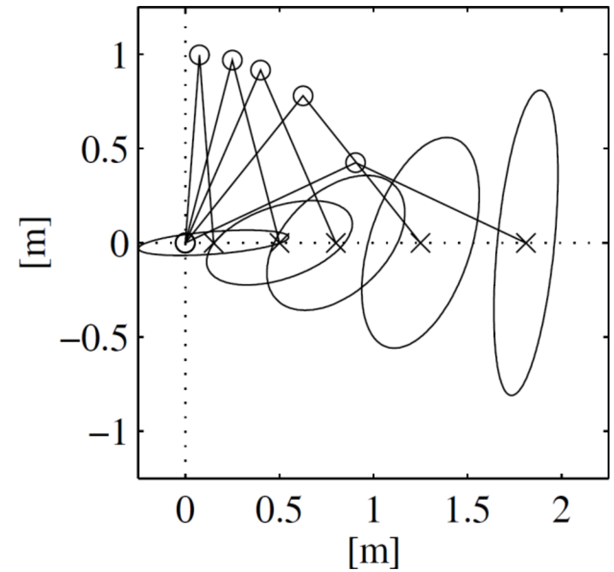
Assume that $\boldsymbol{u}_i \in \mathbb{R}^r$ are eigenvectors and $\lambda_i \in \mathbb{R}$ are eigenvalues of $\boldsymbol{J}\boldsymbol{J}^T$ $(i = 1, \dots, r)$.

- Directions of the principal axes: $\boldsymbol{u}_i$
- Lengths of the principal semi-axes: $\sigma_i = \sqrt{\lambda_i}$ (σ_i are the singular values of $\boldsymbol{J}$)
- Volume is proportional to $\sqrt{\lambda_1\lambda_2\cdots\lambda_r} = \sqrt{\det(\boldsymbol{J}\boldsymbol{J}^T)} \xrightarrow{\text{if } n = r} = |\det(\boldsymbol{J})|$

Velocity Manipulability Ellipsoid

- Along the direction of the **major axis** of the ellipsoid, the end-effector can move at **large velocity**, while along the direction of the **minor axis small end-effector velocities** are obtained.
- The closer the ellipsoid is to a **sphere**, the better the end-effector can move isotropically along all directions of the operational space.
- Velocity manipulability ellipsoid is used to visualize and characterize how close a nonsingular configuration of a robot is to being singular.

Velocity manipulability ellipses for a 2R planar arm (for $l_1 = l_2 = 1$):



Velocity Manipulability Measures

Velocity manipulability measures:

(1) The volume of the ellipsoid (proportional to $\sqrt{\lambda_1\lambda_2 \ldots}$):

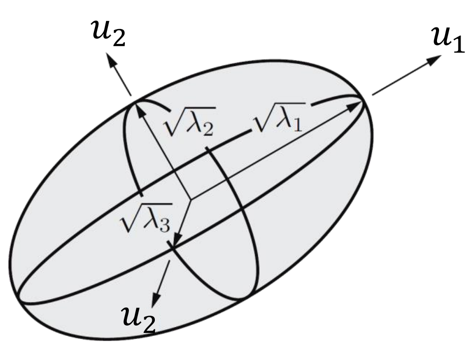
$$w_1(\boldsymbol{\theta}) = \sigma_1\sigma_2 \cdots = \sqrt{\lambda_1\lambda_2 \cdots} = \sqrt{\det(\boldsymbol{J}\boldsymbol{J}^T)} \geq 0 \xrightarrow[\text{(nonredundant)}]{\text{if } n = r} = |\det(\boldsymbol{J})|$$

- As the robot approaches a singularity, $w_1(\boldsymbol{\theta})$ goes to 0.

(2) The ratio of the largest to smallest principal semi-axes:

$$w_2(\boldsymbol{\theta}) = \frac{\sigma_{\max}}{\sigma_{\min}} = \frac{\sqrt{\lambda_{\max}(\boldsymbol{J}\boldsymbol{J}^T)}}{\sqrt{\lambda_{\min}(\boldsymbol{J}\boldsymbol{J}^T)}} = \sqrt{\frac{\lambda_{\max}(\boldsymbol{J}\boldsymbol{J}^T)}{\lambda_{\min}(\boldsymbol{J}\boldsymbol{J}^T)}} \geq 1$$

(w_2 is called <u>condition number</u> of $\boldsymbol{J}$)



(3) The ratio of the largest to smallest eigenvalues:

$$w_3(\boldsymbol{\theta}) = w_2(\boldsymbol{\theta})^2 = \frac{\lambda_{\max}(\boldsymbol{J}\boldsymbol{J}^T)}{\lambda_{\min}(\boldsymbol{J}\boldsymbol{J}^T)} \geq 1$$

- When $w_2(\boldsymbol{\theta})$ or $w_3(\boldsymbol{\theta})$ is low (close to 1), the ellipsoid is nearly spherical or isotropic.
- As the robot approaches a singularity, $w_2(\boldsymbol{\theta})$ or $w_3(\boldsymbol{\theta})$ goes to infinity.

Force Manipulability Ellipsoid

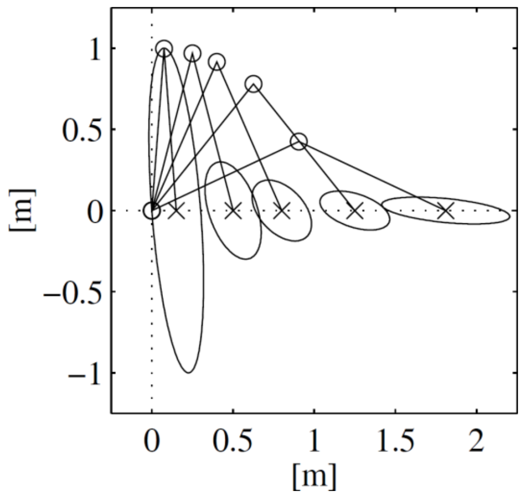
The **force manipulability ellipsoid** corresponds to forces $\boldsymbol{\mathcal{F}}$ generated at the end-effector by joint rates $\boldsymbol{\tau}$ satisfying $\|\boldsymbol{\tau}\| = \boldsymbol{\tau}^T\boldsymbol{\tau} = 1$ (the points on the surface of a sphere).

$$\boldsymbol{\tau} = \boldsymbol{J}^T(\boldsymbol{\theta})\boldsymbol{\mathcal{F}}$$

$$1 = \boldsymbol{\tau}^T\boldsymbol{\tau} = \boldsymbol{\mathcal{F}}^T\boldsymbol{J}\boldsymbol{J}^T\boldsymbol{\mathcal{F}}$$

(the points on the surface of an ellipsoid)

Force manipulability ellipses for a 2R planar arm (for $l_1 = l_2 = 1$):



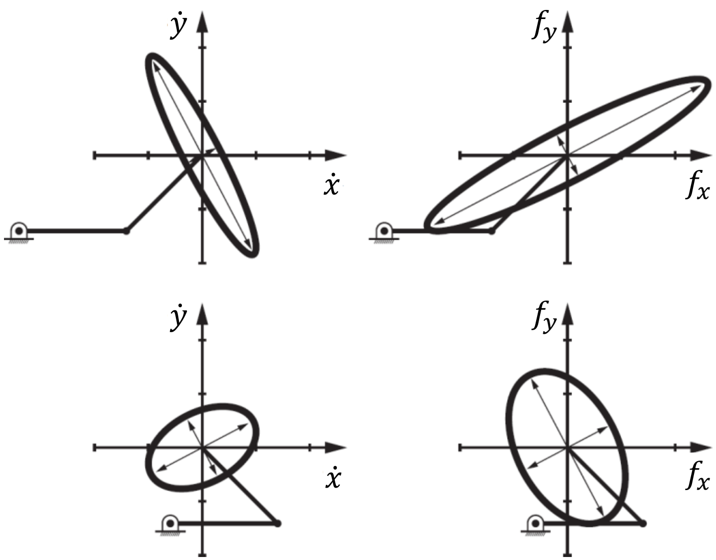
- The principal axes of the force manipulability ellipsoid coincide with the principal axes of the velocity manipulability ellipsoid.
- The lengths of the respective principal semi-axes are in inverse proportion ($1/\sqrt{\lambda_i}$).

Remarks

- According to the concept of **force/velocity duality**, a direction along which it is easy to generate a tip velocity is a direction along which it is difficult to generate a force, and vice versa.

- The <u>product of the volumes</u> of the velocity and force manipulability ellipsoids ($\propto \sqrt{\lambda_1 \lambda_2 \cdots}$ and $\propto 1/\sqrt{\lambda_1 \lambda_2 \cdots}$, respectively) is constant over $\boldsymbol{\theta}$.

For a 2R Planar Robot:



- At a singularity,
  - the velocity manipulability ellipsoid collapses to a line segment (it loses dimension, and its area drops to zero). EE motion capability becomes zero in one (or more) direction(s),
  - the force manipulability ellipsoid becomes infinitely long in a direction orthogonal to the velocity manipulability ellipsoid line segment and skinny in the orthogonal direction (its area goes to infinity). EE can resist infinite force in one (or more) direction(s).

Visualizing Manipulability Ellipsoids

If it is desired to geometrically visualize manipulability in a space of dimension greater than 3, it is worth separating the components of linear velocity (or force) from those of angular velocity (or moment), also avoiding problems due to nonhomogeneous dimensions of the relevant quantities (e.g., m/s vs rad/s).

$$\boldsymbol{J}(\boldsymbol{\theta}) = \begin{bmatrix} \boldsymbol{J}_\omega(\boldsymbol{\theta}) \\ \boldsymbol{J}_v(\boldsymbol{\theta}) \end{bmatrix} \in \mathbb{R}^{6\times n}$$

$\boldsymbol{J}_\omega(\boldsymbol{\theta}) \in \mathbb{R}^{3\times n} \rightarrow$ angular velocity/moment ellipsoids

$\boldsymbol{J}_v(\boldsymbol{\theta}) \in \mathbb{R}^{3\times n} \rightarrow$ linear velocity/force ellipsoids

- When calculating the linear-velocity manipulability ellipsoid, it generally makes more sense to use the body Jacobian $\boldsymbol{J}_b$ or geometric Jacobian $\boldsymbol{J}_g$ instead of the space Jacobian $\boldsymbol{J}_s$, since we are usually interested in the linear velocity of a point at the origin of the end-effector frame rather than that of a point at the origin of the fixed-space frame.