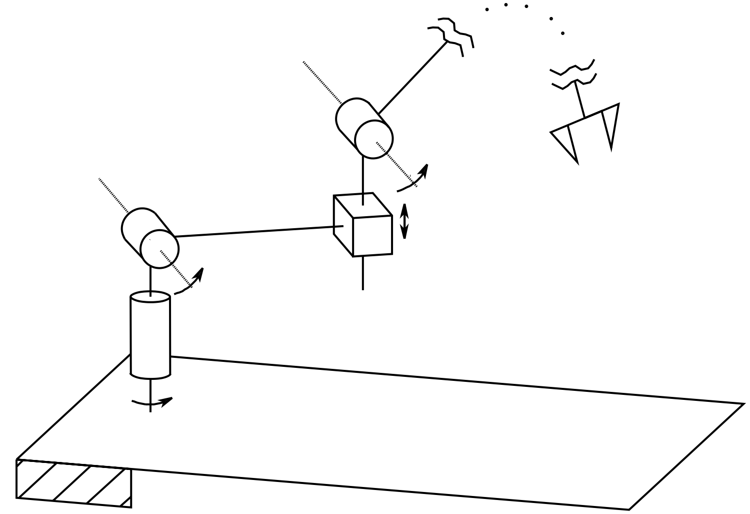
Ch2: Robot Dynamics – Part 1

Dynamic Equations

Dynamic Equations

The dynamic equations (equations of motion) of an open-chain manipulator are a set of 2nd-order ordinary differential equations of the form



$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{h}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$$

$$= \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{c}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) + \boldsymbol{g}(\boldsymbol{\theta})$$

$\boldsymbol{\theta} \in \mathbb{R}^n$: Joint Variables (or joint coordinates or joint positions)

$\boldsymbol{\tau} \in \mathbb{R}^n$: Joint Torques/Forces (applied at the joints by the actuators)

$\boldsymbol{M}(\boldsymbol{\theta}) \in \mathbb{R}^{n\times n}$: Mass Matrix

$\boldsymbol{h}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \in \mathbb{R}^n$: Coriolis, Centripetal, Gravitational, and Frictional Terms

$\boldsymbol{c}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \in \mathbb{R}^n$: Coriolis, Centripetal, and Frictional Terms

$\boldsymbol{g}(\boldsymbol{\theta}) \in \mathbb{R}^n$: Gravitational Terms

Forward & Inverse Dynamics

Forward Dynamics:
Finding the acceleration $\ddot{\boldsymbol{\theta}}$ given the state $\boldsymbol{\theta}$, $\dot{\boldsymbol{\theta}}$, and the joint torques/forces $\boldsymbol{\tau}$:

$$\ddot{\boldsymbol{\theta}} = \boldsymbol{M}^{-1}(\boldsymbol{\theta})(\boldsymbol{\tau} - \boldsymbol{h}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}))$$

Inverse Dynamics:
Finding the joint torques/forces $\boldsymbol{\tau}$ given the state $\boldsymbol{\theta}$, $\dot{\boldsymbol{\theta}}$, and acceleration $\ddot{\boldsymbol{\theta}}$.

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{h}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$$

Two equivalent approaches to derive dynamic equations:

1) **Lagrangian Formulation** (variational, based on energy)
2) **Newton–Euler Formulation**

Lagrangian Formulation

Lagrangian Formulation

- Lagrangian function: $\mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \mathcal{K}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \mathcal{P}(\boldsymbol{q})$

Kinetic Energy | Potential Energy

$\mathcal{L}: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$

$\boldsymbol{q} \in \mathbb{R}^n$: Generalized Independent Coordinates

(Due only to conservative forces such as gravitational energy and energy stored in springs.)

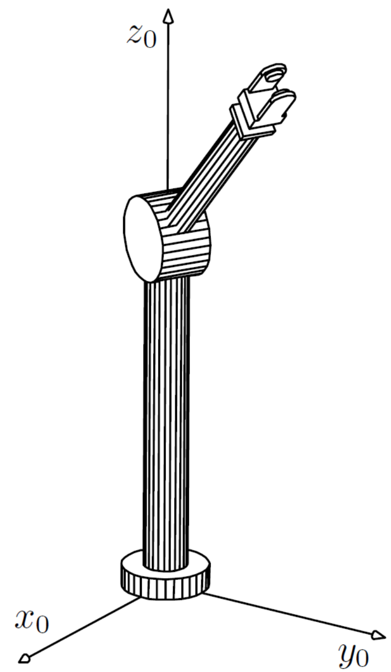
- Equations of Motion: $$\boldsymbol{f} = \frac{d}{dt}\left[\frac{\partial \mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}})}{\partial \dot{\boldsymbol{q}}}\right] - \frac{\partial \mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}})}{\partial \boldsymbol{q}}$$

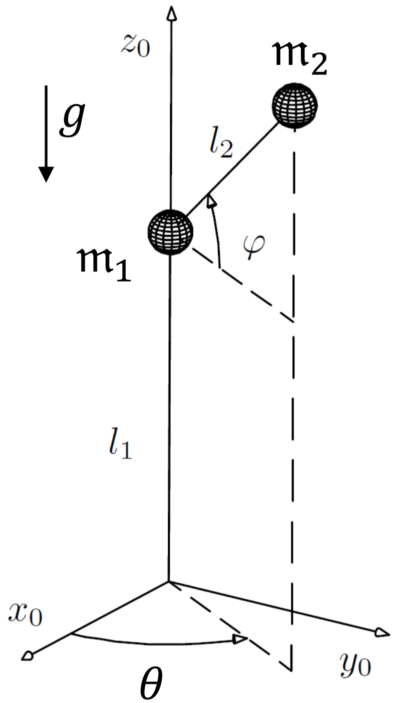
$\boldsymbol{f} \in \mathbb{R}^n$: Generalized (Nonconservative) Forces (e.g., external forces/torques or friction forces) such that $\boldsymbol{f}$ and $\dot{\boldsymbol{q}}$ are dual to each other, i.e., the $\boldsymbol{f}^T\dot{\boldsymbol{q}}$ corresponds to power.

In components: $$f_i = \frac{d}{dt}\left[\frac{\partial \mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}})}{\partial \dot{q}_i}\right] - \frac{\partial \mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}})}{\partial q_i} \qquad i = 1, \cdots, n$$

Example 1

Consider the 1-DOF mechanism. It consists of a rigid link formed by two parts, of lengths l_1 and l_2, whose masses $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are, for simplicity, considered to be concentrated at their respective centers of mass, located at the ends. The angle φ is constant. The mechanism possesses only revolute motion about the z_0 axis, the angle of which is represented by θ. Derive equations of motion for the mechanism moving in the presence of gravity.

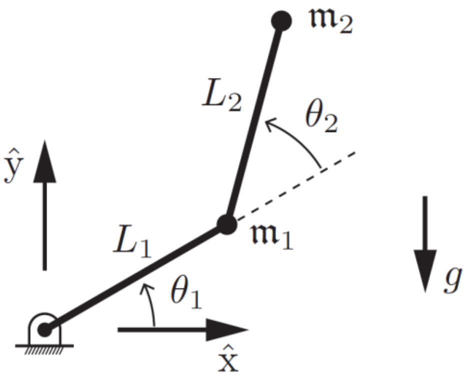




Example 2

Derive equations of motion for a planar 2R open chain moving in the presence of gravity.

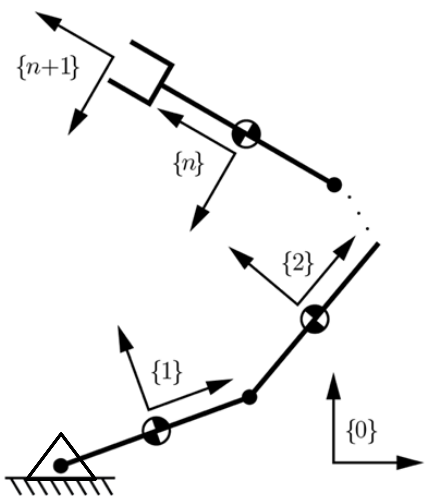
For the sake of simplicity, model the links as point masses $\mathfrak{m}_1$, $\mathfrak{m}_2$ concentrated at the ends of each link.



Note:
For an n-link open-chain manipulator:

Generalized coordinates: $\boldsymbol{\theta} \in \mathbb{R}^n$

Generalized forces: $\boldsymbol{\tau} \in \mathbb{R}^n$



Example 2 (cont.)

$$\tau_1 = \left(\mathfrak{m}_1 L_1^2 + \mathfrak{m}_2(L_1^2 + 2L_1L_2\cos\theta_2 + L_2^2)\right)\ddot{\theta}_1$$
$$+\mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2)\ddot{\theta}_2 - \mathfrak{m}_2L_1L_2\sin\theta_2\left(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2\right)$$
$$+(\mathfrak{m}_1 + \mathfrak{m}_2)L_1 g\cos\theta_1 + \mathfrak{m}_2 gL_2\cos(\theta_1 + \theta_2),$$

$$\tau_2 = \mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2)\ddot{\theta}_1 + \mathfrak{m}_2L_2^2\ddot{\theta}_2 + \mathfrak{m}_2L_1L_2\dot{\theta}_1^2\sin\theta_2$$
$$+\mathfrak{m}_2 gL_2\cos(\theta_1 + \theta_2).$$

We can gather terms together into an equation of the form: $\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{c}\left(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}\right) + \boldsymbol{g}(\boldsymbol{\theta})$

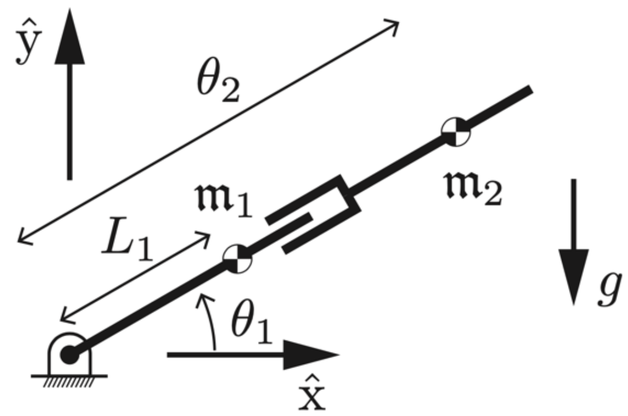
$$\boldsymbol{M}(\boldsymbol{\theta}) = \begin{bmatrix} \mathfrak{m}_1 L_1^2 + \mathfrak{m}_2(L_1^2 + 2L_1L_2\cos\theta_2 + L_2^2) & \mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2) \\ \mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2) & \mathfrak{m}_2L_2^2 \end{bmatrix}$$

$$\boldsymbol{c}\left(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}\right) = \begin{bmatrix} -\mathfrak{m}_2L_1L_2\sin\theta_2\left(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2\right) \\ \mathfrak{m}_2L_1L_2\dot{\theta}_1^2\sin\theta_2 \end{bmatrix}$$

$$\boldsymbol{g}(\boldsymbol{\theta}) = \begin{bmatrix} (\mathfrak{m}_1 + \mathfrak{m}_2)L_1 g\cos\theta_1 + \mathfrak{m}_2 gL_2\cos(\theta_1 + \theta_2) \\ \mathfrak{m}_2 gL_2\cos(\theta_1 + \theta_2) \end{bmatrix}$$

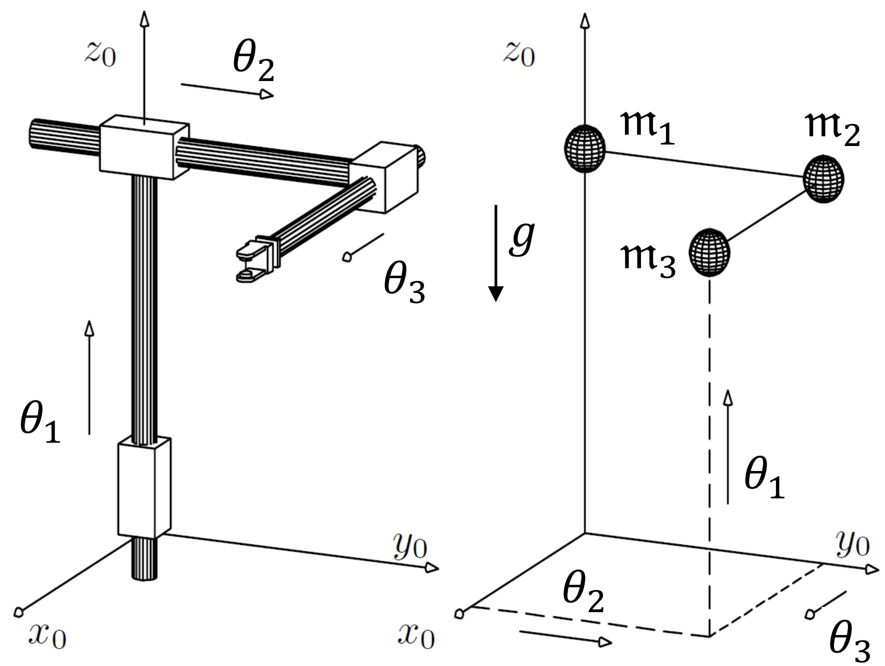
Example 3

Derive equations of motion for a planar RP open chain moving in the presence of gravity.



Example 4

Consider the 3-DOF Cartesian robot manipulator. The manipulator consists of three rigid links mutually orthogonal. The three joints of the robot are prismatic. Derive equations of motion for the robot manipulator moving in the presence of gravity.



Newton–Euler Formulation

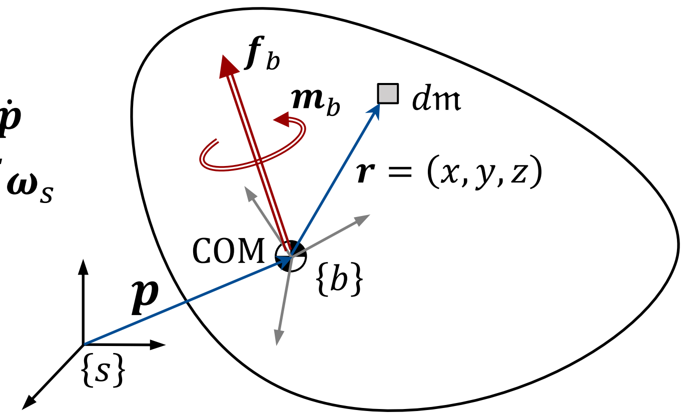
Dynamics of a Single Rigid Body: Method 1

Let assume that the body is moving with a body twist $\boldsymbol{\mathcal{V}}_b = (\boldsymbol{\omega}_b, \boldsymbol{v}_b)$ and $\{b\}$ is at center of mass (COM).

$$\dot{\boldsymbol r} = \boldsymbol v_b + \boldsymbol\omega_b \times \boldsymbol r$$

$$\begin{aligned}\ddot{\boldsymbol r} &= \dot{\boldsymbol v}_b + \frac{d}{dt}\boldsymbol\omega_b \times \boldsymbol r + \boldsymbol\omega_b \times \frac{d}{dt}\boldsymbol r \\ &= \dot{\boldsymbol v}_b + \dot{\boldsymbol\omega}_b \times \boldsymbol r + \boldsymbol\omega_b \times (\boldsymbol v_b + \boldsymbol\omega_b \times \boldsymbol r) \\ &= \dot{\boldsymbol v}_b + [\dot{\boldsymbol\omega}_b]\boldsymbol r + [\boldsymbol\omega_b]\boldsymbol v_b + [\boldsymbol\omega_b]^2\boldsymbol r\end{aligned}$$

$$\boldsymbol R := \boldsymbol R_{sb}$$
$$\boldsymbol v_b = \boldsymbol R^T\dot{\boldsymbol p}$$
$$\boldsymbol\omega_b = \boldsymbol R^T\boldsymbol\omega_s$$



$$d\boldsymbol f = d\mathfrak{m}\,\ddot{\boldsymbol r} = d\mathfrak{m}\,(\dot{\boldsymbol v}_b + [\dot{\boldsymbol\omega}_b]\boldsymbol r + [\boldsymbol\omega_b]\boldsymbol v_b + [\boldsymbol\omega_b]^2\boldsymbol r)$$

At COM: $\int \boldsymbol r d\mathfrak{m} = \int [\boldsymbol r] d\mathfrak{m} = \boldsymbol 0$

$$d\boldsymbol m = \boldsymbol r \times d\boldsymbol f = [\boldsymbol r]d\boldsymbol f$$

$$\begin{bmatrix}\boldsymbol m_b \\ \boldsymbol f_b\end{bmatrix} = \begin{bmatrix}\int d\boldsymbol m \\ \int d\boldsymbol f\end{bmatrix} \begin{matrix}\text{Rotational dynamics} \\ \text{Translational dynamics}\end{matrix} = \begin{bmatrix}\dot{\boldsymbol v}_b \int \cancelto{\boldsymbol 0}{[\boldsymbol r]} d\mathfrak{m} + \int [\boldsymbol r][\dot{\boldsymbol\omega}_b]\boldsymbol r d\mathfrak{m} + [\boldsymbol\omega_b]\boldsymbol v_b \int \cancelto{\boldsymbol 0}{[\boldsymbol r]} d\mathfrak{m} + \int [\boldsymbol r][\boldsymbol\omega_b]^2 \boldsymbol r d\mathfrak{m} \\ \dot{\boldsymbol v}_b \int d\mathfrak{m} + [\dot{\boldsymbol\omega}_b] \int \cancelto{\boldsymbol 0}{\boldsymbol r d\mathfrak{m}} + [\boldsymbol\omega_b]\boldsymbol v_b \int d\mathfrak{m} + [\boldsymbol\omega_b]^2 \int \cancelto{\boldsymbol 0}{\boldsymbol r d\mathfrak{m}}\end{bmatrix}$$

$$= \begin{bmatrix}\boldsymbol I_b\dot{\boldsymbol\omega}_b + [\boldsymbol\omega_b]\boldsymbol I_b\boldsymbol\omega_b \\ \mathfrak{m}(\dot{\boldsymbol v}_b + [\boldsymbol\omega_b]\boldsymbol v_b)\end{bmatrix} \longleftarrow \text{Euler's Equation}$$

$$\boldsymbol I_b = -\int [\boldsymbol r]^2 d\mathfrak{m} \in \mathbb{R}^{3\times3}$$

Inertia Matrix in frame $\{b\}$
(symmetric and positive definite)

Inertia Matrix

$$\boldsymbol{I}_b = -\int [\boldsymbol{r}]^2 d\mathfrak{m} \qquad \boldsymbol{r} = (x, y, z)$$

$$= \begin{bmatrix} \int (y^2+z^2) d\mathfrak{m} & -\int xy d\mathfrak{m} & -\int xz d\mathfrak{m} \\ -\int xy d\mathfrak{m} & \int (x^2+z^2) d\mathfrak{m} & -\int yz d\mathfrak{m} \\ -\int xz d\mathfrak{m} & -\int yz d\mathfrak{m} & \int (x^2+y^2) d\mathfrak{m} \end{bmatrix}$$

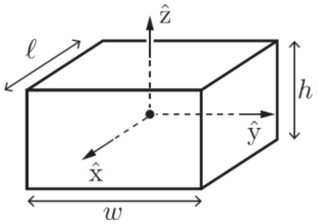
$$= \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix}$$

Note: Inertia matrix in body frame $\{b\}$ ($\boldsymbol{I}_b$) is constant.

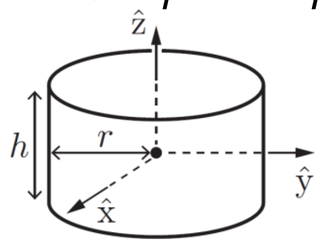
Rotational Kinetic Energy:

$$\mathcal{K} = \frac{1}{2}\boldsymbol{\omega}_b^{\mathrm{T}} \boldsymbol{I}_b \boldsymbol{\omega}_b$$

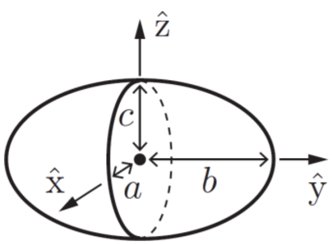
- If the body has uniform density: $d\mathfrak{m} = \rho dV = \rho dxdydz$



$$I_{xx} = \mathfrak{m}(w^2+h^2)/12$$
$$I_{yy} = \mathfrak{m}(\ell^2+h^2)/12$$
$$I_{zz} = \mathfrak{m}(\ell^2+w^2)/12$$



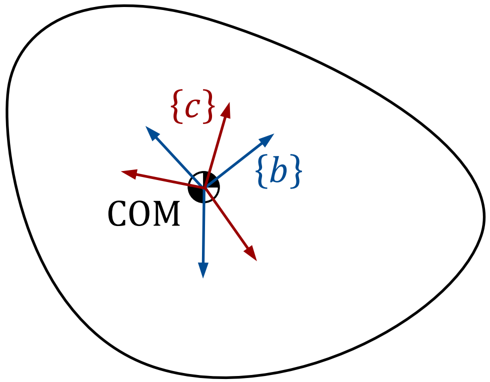
$$I_{xx} = \mathfrak{m}(3r^2+h^2)/12$$
$$I_{yy} = \mathfrak{m}(3r^2+h^2)/12$$
$$I_{zz} = \mathfrak{m}r^2/2$$



$$I_{xx} = \mathfrak{m}(b^2+c^2)/5$$
$$I_{yy} = \mathfrak{m}(a^2+c^2)/5$$
$$I_{zz} = \mathfrak{m}(a^2+b^2)/5$$

Expressing Inertia Matrix $\boldsymbol{I}_b$ in a Rotated Frame

Let $\boldsymbol{I}_c$ be inertia matrix in a rotated frame $\{c\}$ described by $\boldsymbol{R}_{bc}$:



Rotational kinetic energy of the rotating body is independent of the chosen frame:

$$\begin{aligned}\frac{1}{2}\boldsymbol{\omega}_c^{\mathrm{T}}\boldsymbol{I}_c\boldsymbol{\omega}_c &= \frac{1}{2}\boldsymbol{\omega}_b^{\mathrm{T}}\boldsymbol{I}_b\boldsymbol{\omega}_b \\ &= \frac{1}{2}(\boldsymbol{R}_{bc}\boldsymbol{\omega}_c)^{\mathrm{T}}\boldsymbol{I}_b(\boldsymbol{R}_{bc}\boldsymbol{\omega}_c) \\ &= \frac{1}{2}\boldsymbol{\omega}_c^{\mathrm{T}}\left(\boldsymbol{R}_{bc}^{\mathrm{T}}\boldsymbol{I}_b\boldsymbol{R}_{bc}\right)\boldsymbol{\omega}_c\end{aligned}$$

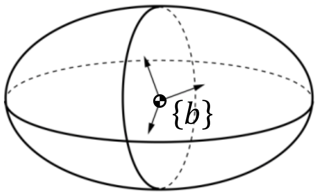
$$\Rightarrow \quad \boldsymbol{I}_c = \boldsymbol{R}_{bc}^{\mathrm{T}}\boldsymbol{I}_b\boldsymbol{R}_{bc}$$

Diagonalizing Inertia Matrix $\boldsymbol{I}_b$

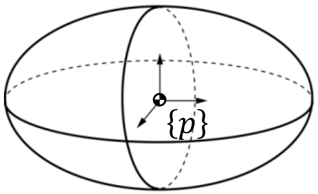
Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ be the eigenvectors of $\boldsymbol{I}_b$ and $\lambda_1, \lambda_2, \lambda_3$ be the corresponding eigenvalues.

- **Principal Axes of Inertia** are in the directions of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ (expressed in $\{b\}$).
- **Principal Moments of Inertia** (about $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$), are $\lambda_1, \lambda_2, \lambda_3 > 0$.

$$\boldsymbol{R}_{bp} = [\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3]$$



$$\boldsymbol{I}_b = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix}$$



$$\boldsymbol{I}_c = \boldsymbol{R}_{bp}^{\mathrm{T}} \boldsymbol{I}_b \boldsymbol{R}_{bp} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$\boldsymbol{\omega}_b = (\omega_x, \omega_y, \omega_z)$$

- If $\{b\}$ is aligned with the principal axes of inertia :

$$\boldsymbol{m}_b = \boldsymbol{I}_b \dot{\boldsymbol{\omega}}_b + [\boldsymbol{\omega}_b] \boldsymbol{I}_b \boldsymbol{\omega}_b = \begin{bmatrix} I_{xx}\dot{\omega}_x + (I_{zz} - I_{yy})\omega_y\omega_z \\ I_{yy}\dot{\omega}_y + (I_{xx} - I_{zz})\omega_x\omega_z \\ I_{zz}\dot{\omega}_z + (I_{yy} - I_{xx})\omega_x\omega_y \end{bmatrix}$$

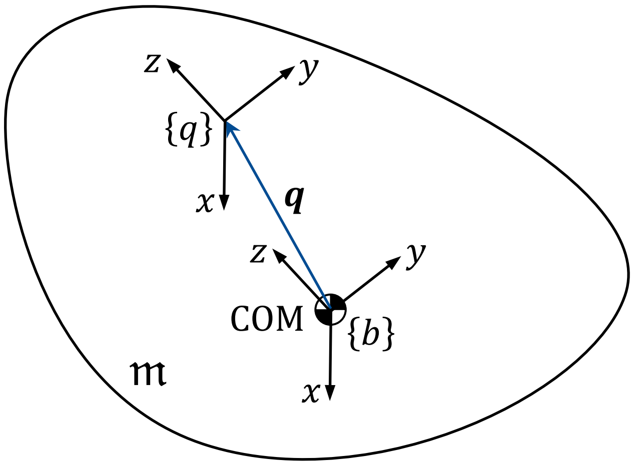
Inertia Matrix: Steiner's Theorem

Inertia matrix $\boldsymbol{I}_q$ about a frame $\{q\}$ aligned with $\{b\}$ (at the center of mass), but at a point $\boldsymbol{q} = \left(q_x, q_y, q_z\right)$ in $\{b\}$:

$$\begin{aligned}\boldsymbol{I}_q &= \boldsymbol{I}_b + \mathfrak{m}\left(\boldsymbol{q}^{\mathrm{T}}\boldsymbol{q}\boldsymbol{I}_3 - \boldsymbol{q}\boldsymbol{q}^{\mathrm{T}}\right) \\ &= \boldsymbol{I}_b + \mathfrak{m}[\boldsymbol{q}]^{\mathrm{T}}[\boldsymbol{q}]\end{aligned}$$

$$\boldsymbol{I}_3 = \mathrm{diag}(1) \in \mathbb{R}^{3\times 3}$$

(identity matrix)



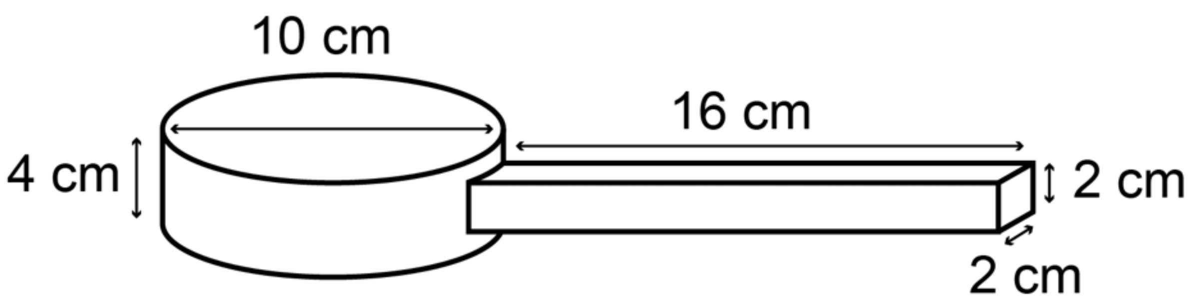
Note: The inertia matrix of a compound/composite body is the <u>sum</u> of their inertias when expressed in a common frame.

Example

A compound object consists of a uniform-density cylinder and a uniform-density rectangular prism. The mass of the cylinder is 2 kg and the mass of the prism is 1 kg. A frame $\{a\}$ is defined at the center of the cylinder, with the x-axis along the prism and the z-axis vertical.

(I) Where is the CM of the compound object in $\{a\}$?

(II) In a frame $\{b\}$ at the CM, aligned with $\{a\}$, what is the inertia of the compound object?



Dynamics of a Single Rigid Body: Method 2
(in World Frame $\{s\}$)

Let $\boldsymbol{f}$ be the net force applied at the center of mass of the rigid body expressed in $\{s\}$ and $\boldsymbol{m}$ be the net moment applied to the rigid body expressed in $\{s\}$.

($\mathfrak{m}\dot{\boldsymbol{p}}$: linear momentum)

($\boldsymbol{I}_s\boldsymbol{\omega}_s$: angular momentum)

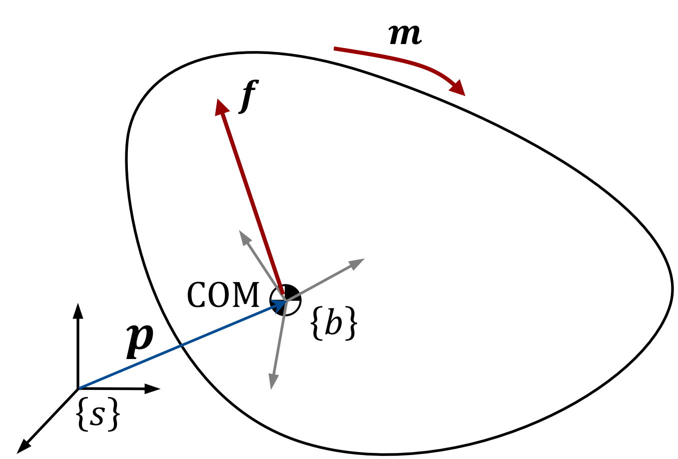
$$\boldsymbol{f} = \frac{d}{dt}(\mathfrak{m}\dot{\boldsymbol{p}}) = \mathfrak{m}\ddot{\boldsymbol{p}}$$

$$\boldsymbol{m} = \frac{d}{dt}(\boldsymbol{I}_s\boldsymbol{\omega}_s) = \frac{d}{dt}(\boldsymbol{R}\boldsymbol{I}_b\boldsymbol{R}^T\boldsymbol{\omega}_s) = \boldsymbol{R}\boldsymbol{I}_b\boldsymbol{R}^T\dot{\boldsymbol{\omega}}_s + \dot{\boldsymbol{R}}\boldsymbol{I}_b\boldsymbol{R}^T\boldsymbol{\omega}_s + \boldsymbol{R}\boldsymbol{I}_b\dot{\boldsymbol{R}}^T\boldsymbol{\omega}_s$$

$$= \boldsymbol{R}\boldsymbol{I}_b\boldsymbol{R}^T\dot{\boldsymbol{\omega}}_s + \underbrace{\dot{\boldsymbol{R}}\boldsymbol{R}^T}_{[\boldsymbol{\omega}_s]}\underbrace{\boldsymbol{R}\boldsymbol{I}_b\boldsymbol{R}^T}_{\boldsymbol{I}_s}\boldsymbol{\omega}_s + \underbrace{\boldsymbol{R}\boldsymbol{I}_b\boldsymbol{R}^T}_{\boldsymbol{I}_s}\cancelto{\mathbf{0}}{\underbrace{\boldsymbol{R}\dot{\boldsymbol{R}}^T}_{-[\boldsymbol{\omega}_s]}\boldsymbol{\omega}_s}$$

$$= \boldsymbol{I}_s\dot{\boldsymbol{\omega}}_s + [\boldsymbol{\omega}_s]\boldsymbol{I}_s\boldsymbol{\omega}_s$$

$$\begin{bmatrix}\boldsymbol{m}\\ \boldsymbol{f}\end{bmatrix} = \begin{bmatrix}\boldsymbol{I}_s\dot{\boldsymbol{\omega}}_s + [\boldsymbol{\omega}_s]\boldsymbol{I}_s\boldsymbol{\omega}_s\\ \mathfrak{m}\ddot{\boldsymbol{p}}\end{bmatrix}$$



$\boldsymbol{R} := \boldsymbol{R}_{sb}$

$\boldsymbol{I}_s = \boldsymbol{R}\boldsymbol{I}_b\boldsymbol{R}^T$

($\boldsymbol{I}_s$: Inertia matrix of body about COM in a frame aligned with $\{s\}$)

Dynamics of a Single Rigid Body: Method 2
(in Body Frame $\{b\}$)

Let $\boldsymbol{f}_b$ be the net force applied at the center of mass of the rigid body expressed in $\{b\}$ and $\boldsymbol{m}_b$ be the net moment applied to the rigid body expressed in $\{b\}$.

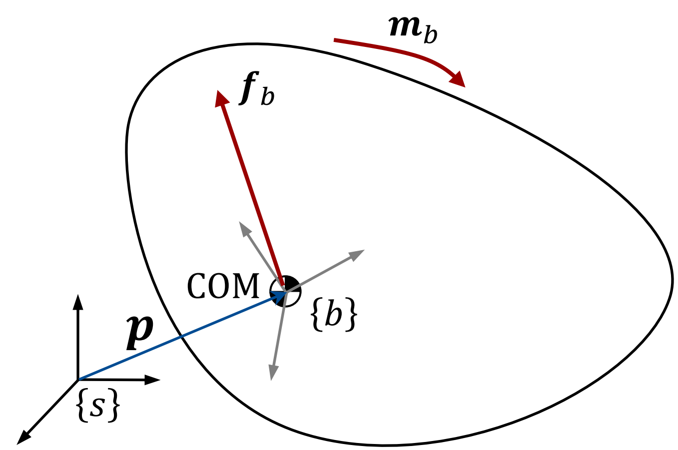
$$\boldsymbol{f} = \frac{d}{dt}(\mathrm{m}\dot{\boldsymbol{p}}) = \frac{d}{dt}(\mathrm{m}\boldsymbol{R}\boldsymbol{v}_b) = \mathrm{m}\boldsymbol{R}\dot{\boldsymbol{v}}_b + \mathrm{m}\dot{\boldsymbol{R}}\boldsymbol{v}_b$$

$$\boldsymbol{m} = \frac{d}{dt}(\boldsymbol{I}_s\boldsymbol{\omega}_s) = \frac{d}{dt}(\boldsymbol{R}\boldsymbol{I}_b\boldsymbol{R}^T\boldsymbol{\omega}_s) = \frac{d}{dt}(\boldsymbol{R}\boldsymbol{I}_b\underbrace{\boldsymbol{R}^T\boldsymbol{R}}_{\boldsymbol{I}_3}\boldsymbol{\omega}_b) = \frac{d}{dt}(\boldsymbol{R}\boldsymbol{I}_b\boldsymbol{\omega}_b) = \boldsymbol{R}\boldsymbol{I}_b\dot{\boldsymbol{\omega}}_b + \dot{\boldsymbol{R}}\boldsymbol{I}_b\boldsymbol{\omega}_b$$

$$\boldsymbol{f}_b = \boldsymbol{R}^T\boldsymbol{f} = \underbrace{\boldsymbol{R}^T\boldsymbol{R}}_{\boldsymbol{I}_3}\mathrm{m}\dot{\boldsymbol{v}}_b + \underbrace{\boldsymbol{R}^T\dot{\boldsymbol{R}}}_{[\boldsymbol{\omega}_b]}\mathrm{m}\boldsymbol{v}_b$$

$$\boldsymbol{m}_b = \boldsymbol{R}^T\boldsymbol{m} = \underbrace{\boldsymbol{R}^T\boldsymbol{R}}_{\boldsymbol{I}_3}\boldsymbol{I}_b\dot{\boldsymbol{\omega}}_b + \underbrace{\boldsymbol{R}^T\dot{\boldsymbol{R}}}_{[\boldsymbol{\omega}_b]}\boldsymbol{I}_b\boldsymbol{\omega}_b$$

$$\begin{bmatrix}\boldsymbol{m}_b\\ \boldsymbol{f}_b\end{bmatrix} = \begin{bmatrix}\boldsymbol{I}_b\dot{\boldsymbol{\omega}}_b + [\boldsymbol{\omega}_b]\boldsymbol{I}_b\boldsymbol{\omega}_b\\ \mathrm{m}(\dot{\boldsymbol{v}}_b + [\boldsymbol{\omega}_b]\boldsymbol{v}_b)\end{bmatrix}$$



$\boldsymbol{R} := \boldsymbol{R}_{sb}$

$\boldsymbol{v}_b = \boldsymbol{R}^T\dot{\boldsymbol{p}}$ $\quad$ $\boldsymbol{f}_b = \boldsymbol{R}^T\boldsymbol{f}$

$\boldsymbol{\omega}_b = \boldsymbol{R}^T\boldsymbol{\omega}_s$ $\quad$ $\boldsymbol{m}_b = \boldsymbol{R}^T\boldsymbol{m}$ $\quad$ $\boldsymbol{I}_s = \boldsymbol{R}\boldsymbol{I}_b\boldsymbol{R}^T$

Twist–Wrench Formulation

$$\underbrace{\boldsymbol{\mathcal{F}}_b}_{\text{Body Wrench}} = \begin{bmatrix} \boldsymbol{m}_b \\ \boldsymbol{f}_b \end{bmatrix} = \begin{bmatrix} \boldsymbol{I}_b \dot{\boldsymbol{\omega}}_b + [\boldsymbol{\omega}_b]\boldsymbol{I}_b\boldsymbol{\omega}_b \\ \mathfrak{m}(\dot{\boldsymbol{v}}_b + [\boldsymbol{\omega}_b]\boldsymbol{v}_b) \end{bmatrix} = \begin{bmatrix} \boldsymbol{I}_b & \boldsymbol{0} \\ \boldsymbol{0} & \mathfrak{m}\boldsymbol{I}_3 \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\omega}}_b \\ \dot{\boldsymbol{v}}_b \end{bmatrix} + \begin{bmatrix} [\boldsymbol{\omega}_b] & \boldsymbol{0} \\ \boldsymbol{0} & [\boldsymbol{\omega}_b] \end{bmatrix} \begin{bmatrix} \boldsymbol{I}_b & \boldsymbol{0} \\ \boldsymbol{0} & \mathfrak{m}\boldsymbol{I}_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{\omega}_b \\ \boldsymbol{v}_b \end{bmatrix}$$

$$= \begin{bmatrix} \boldsymbol{I}_b & \boldsymbol{0} \\ \boldsymbol{0} & \mathfrak{m}\boldsymbol{I}_3 \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\omega}}_b \\ \dot{\boldsymbol{v}}_b \end{bmatrix} + \begin{bmatrix} [\boldsymbol{\omega}_b] & [\boldsymbol{v}_b] \\ \boldsymbol{0} & [\boldsymbol{\omega}_b] \end{bmatrix} \begin{bmatrix} \boldsymbol{I}_b & \boldsymbol{0} \\ \boldsymbol{0} & \mathfrak{m}\boldsymbol{I}_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{\omega}_b \\ \boldsymbol{v}_b \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \boldsymbol{I}_b & \boldsymbol{0} \\ \boldsymbol{0} & \mathfrak{m}\boldsymbol{I}_3 \end{bmatrix}}_{\boldsymbol{\mathcal{G}}_b \in \mathbb{R}^{6\times 6}} \begin{bmatrix} \dot{\boldsymbol{\omega}}_b \\ \dot{\boldsymbol{v}}_b \end{bmatrix} - \underbrace{\begin{bmatrix} [\boldsymbol{\omega}_b] & \boldsymbol{0} \\ [\boldsymbol{v}_b] & [\boldsymbol{\omega}_b] \end{bmatrix}^{\mathrm{T}}}_{[\mathrm{ad}_{\boldsymbol{\mathcal{V}}_b}]} \begin{bmatrix} \boldsymbol{I}_b & \boldsymbol{0} \\ \boldsymbol{0} & \mathfrak{m}\boldsymbol{I}_3 \end{bmatrix} \underbrace{\begin{bmatrix} \boldsymbol{\omega}_b \\ \boldsymbol{v}_b \end{bmatrix}}_{\boldsymbol{\mathcal{V}}_b}$$

$\boldsymbol{\mathcal{G}}_b \in \mathbb{R}^{6\times 6}$: Spatial Inertia Matrix (symmetric & positive definite)

$[\mathrm{ad}_{\boldsymbol{\mathcal{V}}_b}]$

$\boldsymbol{\mathcal{V}}_b$: Body Twist

$$\boldsymbol{\mathcal{F}}_b = \boldsymbol{\mathcal{G}}_b \dot{\boldsymbol{\mathcal{V}}}_b - [\mathrm{ad}_{\boldsymbol{\mathcal{V}}_b}]^{\mathrm{T}} \boldsymbol{\mathcal{G}}_b \boldsymbol{\mathcal{V}}_b$$ (Inverse Dynamics of Rigid Body)

$$\dot{\boldsymbol{\mathcal{V}}}_b = \boldsymbol{\mathcal{G}}_b^{-1}\left(\boldsymbol{\mathcal{F}}_b + [\mathrm{ad}_{\boldsymbol{\mathcal{V}}_b}]^{\mathrm{T}} \boldsymbol{\mathcal{G}}_b \boldsymbol{\mathcal{V}}_b\right)$$ (Forward Dynamics of Rigid Body)

$$\text{Total Kinetic Energy} = \frac{1}{2}\boldsymbol{\omega}_b^{\mathrm{T}}\boldsymbol{I}_b\boldsymbol{\omega}_b + \frac{1}{2}\mathfrak{m}\boldsymbol{v}_b^{\mathrm{T}}\boldsymbol{v}_b = \frac{1}{2}\boldsymbol{\mathcal{V}}_b^{\mathrm{T}}\boldsymbol{\mathcal{G}}_b\boldsymbol{\mathcal{V}}_b$$

Lie Bracket of Two Twists

Given two twists $\mathcal{V}_1 = (\boldsymbol{\omega}_1, \boldsymbol{v}_1) \in \mathbb{R}^6$ and $\mathcal{V}_2 = (\boldsymbol{\omega}_2, \boldsymbol{v}_2) \in \mathbb{R}^6$, the Lie Bracket of $\mathcal{V}_1$ and $\mathcal{V}_2$ is defined as $[\mathrm{ad}_{\mathcal{V}_1}]\mathcal{V}_2 \in \mathbb{R}^6$ where

$$[\mathrm{ad}_{\mathcal{V}}] = \begin{bmatrix} [\boldsymbol{\omega}] & \mathbf{0} \\ [\boldsymbol{v}] & [\boldsymbol{\omega}] \end{bmatrix} \in \mathbb{R}^{6\times 6}, \qquad \mathcal{V} = (\boldsymbol{\omega}, \boldsymbol{v})$$

This is generalization of the cross product to two twists $\mathcal{V}_1$ and $\mathcal{V}_2$.

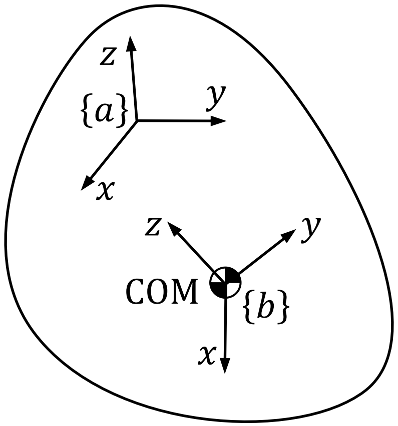
Dynamics in Other Frames

Kinetic energy of the rigid body is independent of the frame of representation:

$$\frac{1}{2}\mathcal{V}_a^{\mathrm{T}}\mathcal{G}_a\mathcal{V}_a = \frac{1}{2}\mathcal{V}_b^{\mathrm{T}}\mathcal{G}_b\mathcal{V}_b = \frac{1}{2}\left([\mathrm{Ad}_{T_{ba}}]\mathcal{V}_a\right)^{\mathrm{T}}\mathcal{G}_b[\mathrm{Ad}_{T_{ba}}]\mathcal{V}_a = \frac{1}{2}\mathcal{V}_a^{\mathrm{T}}\underbrace{[\mathrm{Ad}_{T_{ba}}]^{\mathrm{T}}\mathcal{G}_b[\mathrm{Ad}_{T_{ba}}]}_{\mathcal{G}_a}\mathcal{V}_a$$

$$\Rightarrow \quad \mathcal{G}_a = [\mathrm{Ad}_{T_{ba}}]^{\mathrm{T}}\mathcal{G}_b[\mathrm{Ad}_{T_{ba}}]$$

This is a generalization of Steiner's theorem.



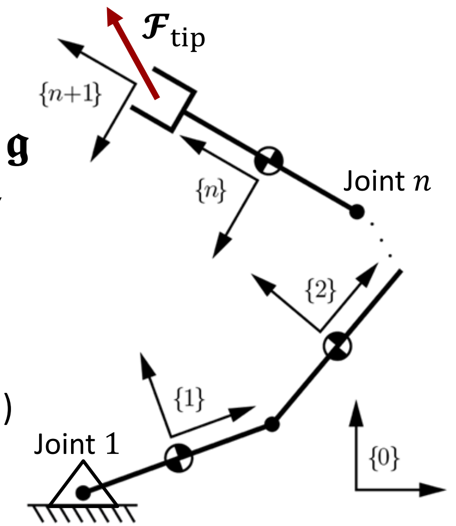
$$\begin{aligned}
\mathcal{F}_a &= [\mathrm{Ad}_{T_{ba}}]^T\mathcal{F}_b \\
&= [\mathrm{Ad}_{T_{ba}}]^T\mathcal{G}_b\dot{\mathcal{V}}_b - [\mathrm{Ad}_{T_{ba}}]^T[\mathrm{ad}_{\mathcal{V}_b}]^T\mathcal{G}_b\mathcal{V}_b \\
&= [\mathrm{Ad}_{T_{ba}}]^T\mathcal{G}_b[\mathrm{Ad}_{T_{ba}}]\dot{\mathcal{V}}_a - [\mathrm{Ad}_{T_{ba}}]^T[\mathrm{ad}_{\mathcal{V}_b}]^T\mathcal{G}_b[\mathrm{Ad}_{T_{ba}}]\mathcal{V}_a \\
&= \mathcal{G}_a\dot{\mathcal{V}}_a - [\mathrm{ad}_{\mathcal{V}_a}]^T\mathcal{G}_a\mathcal{V}_a
\end{aligned}$$

$$\Rightarrow \quad \mathcal{F}_a = \mathcal{G}_a\dot{\mathcal{V}}_a - [\mathrm{ad}_{\mathcal{V}_a}]^T\mathcal{G}_a\mathcal{V}_a$$

The form of the equations of motion is independent of the frame of representation.

Dynamics of an Open Chain Manipulator

Consider an n-link open chain manipulator connected by 1 DOF joints. Attach a frame $\{0\}$ to the base, frames $\{1\}$ to $\{n\}$ to the centers of mass of links 1 to n, and a frame $\{n+1\}$ at the end-effector, fixed in the frame $\{n\}$.



$\mathcal{G}_i \in \mathbb{R}^{6\times 6}$: spatial inertia matrix of link i in $\{i\}$: $\mathcal{G}_i = \begin{bmatrix} \boldsymbol{I}_i & 0 \\ 0 & \mathfrak{m}_i \boldsymbol{I}_3 \end{bmatrix}$

$\boldsymbol{M}_{i,i-1} \in SE(3)$: $\{i-1\}$ in $\{i\}$ at home configuration ($\boldsymbol{\theta} = \mathbf{0}$).

For given $\boldsymbol{M}_{i-1,i}$: $\boldsymbol{M}_{i,i-1} = \left(\boldsymbol{M}_{i-1,i}\right)^{-1}$. $\quad (\boldsymbol{M}_{0,i} = \boldsymbol{M}_{0,1}\boldsymbol{M}_{1,2} \dots \boldsymbol{M}_{i-1,i})$

$\mathcal{A}_i \in \mathbb{R}^6$: screw axis of joint i in $\{i\}$. $\quad \mathcal{A}_i = \left[\mathrm{Ad}_{\boldsymbol{M}_{0,i}^{-1}}\right] \mathcal{S}_i = \left[\mathrm{Ad}_{\boldsymbol{M}_{0,i}}\right]^{-1} \mathcal{S}_i$

($\mathcal{S}_i$ is screw axis of joint i in $\{0\}$)

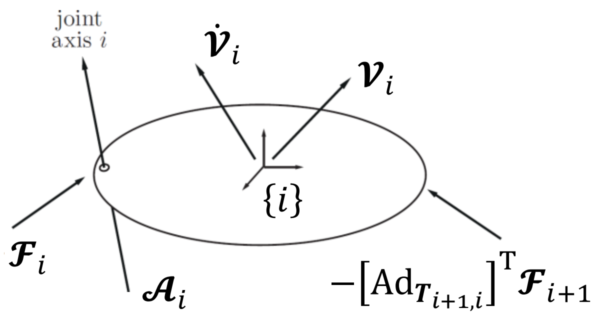
$\mathcal{V}_i = (\boldsymbol{\omega}_i, \boldsymbol{v}_i) \in \mathbb{R}^6$: twist of link i in $\{i\}$.

$\mathcal{V}_0 = (\boldsymbol{\omega}_0, \boldsymbol{v}_0) = \mathbf{0}$ (for a fixed-base manipulator)

$\dot{\mathcal{V}}_0 = (\dot{\boldsymbol{\omega}}_0, \dot{\boldsymbol{v}}_0) = (\mathbf{0}, -\mathfrak{g})$ $\quad \mathfrak{g} \in \mathbb{R}^3$ gravity vector in $\{0\}$

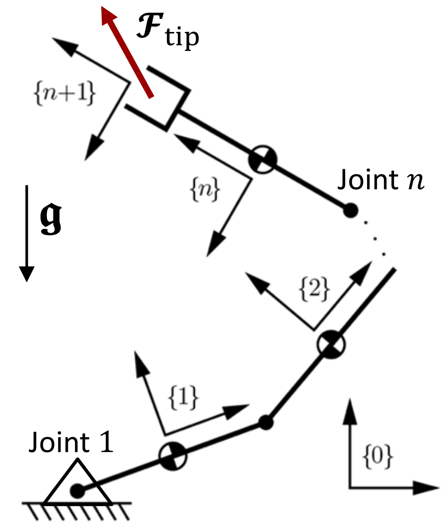
$\mathcal{F}_i = (\boldsymbol{m}_i, \boldsymbol{f}_i) \in \mathbb{R}^6$: wrench at joint i in $\{i\}$.

$\mathcal{F}_{n+1} = \mathcal{F}_{\text{tip}} \in \mathbb{R}^6$: wrench applied to the environment by end-effector in $\{i+1\}$.



Recursive Newton-Euler Inverse Dynamics Algorithm

Forward Iterations: Determining twists $\mathcal{V}_i$ and accelerations $\dot{\mathcal{V}}_i$ of each link by moving outward from the base to the tip. Given $\boldsymbol{\theta}$, $\dot{\boldsymbol{\theta}}$, $\ddot{\boldsymbol{\theta}}$, for $i = 1$ to n, find



$$\boldsymbol{T}_{i,i-1} = e^{-[\mathcal{A}_i]\theta_i}\boldsymbol{M}_{i,i-1} \in SE(3) \qquad (\boldsymbol{T}_{n+1,n} = \boldsymbol{M}_{n+1,n})$$

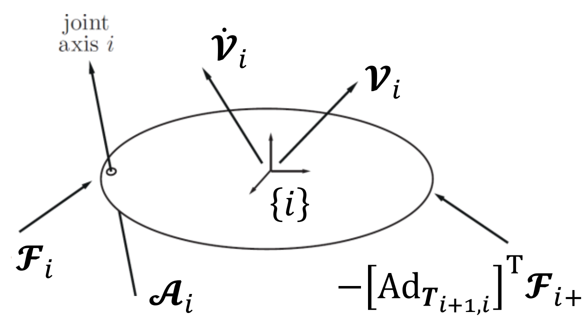
$$\mathcal{V}_i = [\mathrm{Ad}_{\boldsymbol{T}_{i,i-1}}]\mathcal{V}_{i-1} + \mathcal{A}_i\dot{\theta}_i$$

$$\begin{aligned}\dot{\mathcal{V}}_i &= [\mathrm{Ad}_{\boldsymbol{T}_{i,i-1}}]\dot{\mathcal{V}}_{i-1} + \frac{d}{dt}([\mathrm{Ad}_{\boldsymbol{T}_{i,i-1}}])\mathcal{V}_{i-1} + \mathcal{A}_i\ddot{\theta}_i \\ &= [\mathrm{Ad}_{\boldsymbol{T}_{i,i-1}}]\dot{\mathcal{V}}_{i-1} + [\mathrm{ad}_{\mathcal{V}_i}]\mathcal{A}_i\dot{\theta}_i + \mathcal{A}_i\ddot{\theta}_i\end{aligned}$$

Backward Iterations: Determining wrenches $\mathcal{F}_i$ experienced by each link, and then, the joint torques/forces $\boldsymbol{\tau}_i$ by moving inward from the tip to the base. For $i = n$ to 1, find

$$\mathcal{F}_i = [\mathrm{Ad}_{\boldsymbol{T}_{i+1,i}}]^{\mathrm{T}}\mathcal{F}_{i+1} + \mathcal{G}_i\dot{\mathcal{V}}_i - [\mathrm{ad}_{\mathcal{V}_i}]^{\mathrm{T}}\mathcal{G}_i\mathcal{V}_i$$

$$\boldsymbol{\tau}_i = \mathcal{F}_i^{\mathrm{T}}\mathcal{A}_i$$



Dynamic Equations in Closed Form

Let define some stacked vectors and matrices: $\boldsymbol{\mathcal{V}} = \begin{bmatrix} \mathcal{V}_1 \\ \vdots \\ \mathcal{V}_n \end{bmatrix} \in \mathbb{R}^{6n}, \boldsymbol{\mathcal{F}} = \begin{bmatrix} \mathcal{F}_1 \\ \vdots \\ \mathcal{F}_n \end{bmatrix} \in \mathbb{R}^{6n},$

$$\boldsymbol{\mathcal{V}}_{\text{base}} = \begin{bmatrix} [\mathrm{Ad}_{T_{10}}]\,\mathcal{V}_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{6n}, \dot{\boldsymbol{\mathcal{V}}}_{\text{base}} = \begin{bmatrix} [\mathrm{Ad}_{T_{10}}]\,\dot{\mathcal{V}}_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{6n}, \overline{\boldsymbol{\mathcal{F}}}_{\text{tip}} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ [\mathrm{Ad}^{\mathrm{T}}_{T_{n+1,n}}]\,\mathcal{F}_{n+1} \end{bmatrix} \in \mathbb{R}^{6n}$$

$$[\mathrm{ad}_{\boldsymbol{\mathcal{V}}}] = \begin{bmatrix} [\mathrm{ad}_{\mathcal{V}_1}] & 0 & \cdots & 0 \\ 0 & [\mathrm{ad}_{\mathcal{V}_2}] & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & [\mathrm{ad}_{\mathcal{V}_n}] \end{bmatrix} \in \mathbb{R}^{6n\times 6n}, \qquad \boldsymbol{\mathcal{A}} = \begin{bmatrix} \mathcal{A}_1 & 0 & \cdots & 0 \\ 0 & \mathcal{A}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \mathcal{A}_n \end{bmatrix} \in \mathbb{R}^{6n\times n},$$

$$[\mathrm{ad}_{\boldsymbol{\mathcal{A}}\dot{\boldsymbol{\theta}}}] = \begin{bmatrix} [\mathrm{ad}_{\mathcal{A}_1\dot{\theta}_1}] & 0 & \cdots & 0 \\ 0 & [\mathrm{ad}_{\mathcal{A}_2\dot{\theta}_2}] & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & [\mathrm{ad}_{\mathcal{A}_n\dot{\theta}_n}] \end{bmatrix} \in \mathbb{R}^{6n\times 6n}, \qquad \boldsymbol{\mathcal{G}} = \begin{bmatrix} \mathcal{G}_1 & 0 & \cdots & 0 \\ 0 & \mathcal{G}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \mathcal{G}_n \end{bmatrix} \in \mathbb{R}^{6n\times 6n},$$

Note: $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{G}}$ are constant block-diagonal matrices. $\boldsymbol{\mathcal{A}}$ contains only the kinematic parameters while $\boldsymbol{\mathcal{G}}$ contains only the mass and inertial parameters for each link.

Dynamic Equations in Closed Form

$$\boldsymbol{\mathcal{W}}(\boldsymbol{\theta}) = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ [\mathrm{Ad}_{\boldsymbol{T}_{21}}] & 0 & \cdots & 0 & 0 \\ 0 & [\mathrm{Ad}_{\boldsymbol{T}_{32}}] & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & [\mathrm{Ad}_{\boldsymbol{T}_{n,n-1}}] & 0 \end{bmatrix} \in \mathbb{R}^{6n\times 6n}$$

With these definitions, the recursive inverse dynamics algorithm can be assembled into a set of matrix equations:

$$\begin{aligned}
\boldsymbol{\mathcal{V}} &= \boldsymbol{\mathcal{W}}(\boldsymbol{\theta})\boldsymbol{\mathcal{V}} + \boldsymbol{\mathcal{A}}\dot{\boldsymbol{\theta}} + \boldsymbol{\mathcal{V}}_{\text{base}} \\
\dot{\boldsymbol{\mathcal{V}}} &= \boldsymbol{\mathcal{W}}(\boldsymbol{\theta})\dot{\boldsymbol{\mathcal{V}}} + \boldsymbol{\mathcal{A}}\ddot{\boldsymbol{\theta}} - [\mathrm{ad}_{\boldsymbol{\mathcal{A}}\dot{\boldsymbol{\theta}}}](\boldsymbol{\mathcal{W}}(\boldsymbol{\theta})\boldsymbol{\mathcal{V}} + \boldsymbol{\mathcal{V}}_{\text{base}}) + \dot{\boldsymbol{\mathcal{V}}}_{\text{base}} \\
\boldsymbol{\mathcal{F}} &= \boldsymbol{\mathcal{W}}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{\mathcal{F}} + \boldsymbol{\mathcal{G}}\dot{\boldsymbol{\mathcal{V}}} - [\mathrm{ad}_{\boldsymbol{\mathcal{V}}}]^{\mathrm{T}}\boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{V}} + \bar{\boldsymbol{\mathcal{F}}}_{\text{tip}}, \\
\boldsymbol{\tau} &= \boldsymbol{\mathcal{A}}^{\mathrm{T}}\boldsymbol{\mathcal{F}}
\end{aligned}$$

$$\Downarrow$$

$$\begin{aligned}
\boldsymbol{\mathcal{V}} &= \left(\boldsymbol{I}_{6n} - \boldsymbol{\mathcal{W}}(\boldsymbol{\theta})\right)^{-1}\left(\boldsymbol{\mathcal{A}}\dot{\boldsymbol{\theta}} + \boldsymbol{\mathcal{V}}_{\text{base}}\right) \\
\dot{\boldsymbol{\mathcal{V}}} &= \left(\boldsymbol{I}_{6n} - \boldsymbol{\mathcal{W}}(\boldsymbol{\theta})\right)^{-1}\left(\boldsymbol{\mathcal{A}}\ddot{\boldsymbol{\theta}} - [\mathrm{ad}_{\boldsymbol{\mathcal{A}}\dot{\boldsymbol{\theta}}}](\boldsymbol{\mathcal{W}}(\boldsymbol{\theta})\boldsymbol{\mathcal{V}} + \boldsymbol{\mathcal{V}}_{\text{base}}) + \dot{\boldsymbol{\mathcal{V}}}_{\text{base}}\right) \\
\boldsymbol{\mathcal{F}} &= \left(\boldsymbol{I}_{6n} - \boldsymbol{\mathcal{W}}(\boldsymbol{\theta})\right)^{-T}\left(\boldsymbol{\mathcal{G}}\dot{\boldsymbol{\mathcal{V}}} - [\mathrm{ad}_{\boldsymbol{\mathcal{V}}}]^{\mathrm{T}}\boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{V}} + \bar{\boldsymbol{\mathcal{F}}}_{\text{tip}}\right), \\
\boldsymbol{\tau} &= \boldsymbol{\mathcal{A}}^{\mathrm{T}}\boldsymbol{\mathcal{F}}
\end{aligned}$$

Dynamic Equations in Closed Form

The matrix $\boldsymbol{\mathcal{W}}(\boldsymbol{\theta}) \in \mathbb{R}^{6n\times 6n}$ has the property that $\boldsymbol{\mathcal{W}}(\boldsymbol{\theta})^n = \mathbf{0}$ (such a matrix is said to be Nilpotent of order n), and

$$(\boldsymbol{I}_{6n} - \boldsymbol{\mathcal{W}}(\boldsymbol{\theta}))^{-1} = \boldsymbol{I}_{6n} + \boldsymbol{\mathcal{W}}(\boldsymbol{\theta}) + \cdots + \boldsymbol{\mathcal{W}}^{n-1}(\boldsymbol{\theta})$$

$$= \begin{bmatrix} \boldsymbol{I}_n & 0 & 0 & \cdots & 0 \\ [\mathrm{Ad}_{\boldsymbol{T}_{21}}] & \boldsymbol{I}_n & 0 & \cdots & 0 \\ [\mathrm{Ad}_{\boldsymbol{T}_{31}}] & [\mathrm{Ad}_{\boldsymbol{T}_{32}}] & \boldsymbol{I}_n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ [\mathrm{Ad}_{\boldsymbol{T}_{n1}}] & [\mathrm{Ad}_{\boldsymbol{T}_{n2}}] & [\mathrm{Ad}_{\boldsymbol{T}_{n3}}] & \cdots & \boldsymbol{I}_n \end{bmatrix} \in \mathbb{R}^{6n\times 6n}$$

By defining $\boldsymbol{\mathcal{L}}(\boldsymbol{\theta}) = (\boldsymbol{I}_{6n} - \boldsymbol{\mathcal{W}}(\boldsymbol{\theta}))^{-1}$, the equations can be reorganized as:

$$\begin{aligned} \boldsymbol{\mathcal{V}} &= \boldsymbol{\mathcal{L}}(\boldsymbol{\theta})\left(\boldsymbol{\mathcal{A}}\dot{\boldsymbol{\theta}} + \boldsymbol{\mathcal{V}}_{\text{base}}\right) \\ \dot{\boldsymbol{\mathcal{V}}} &= \boldsymbol{\mathcal{L}}(\boldsymbol{\theta})\left(\boldsymbol{\mathcal{A}}\ddot{\boldsymbol{\theta}} - [\mathrm{ad}_{\boldsymbol{\mathcal{A}}\dot{\boldsymbol{\theta}}}](\boldsymbol{\mathcal{W}}(\boldsymbol{\theta})\boldsymbol{\mathcal{V}} + \boldsymbol{\mathcal{V}}_{\text{base}}) + \dot{\boldsymbol{\mathcal{V}}}_{\text{base}}\right) \\ \boldsymbol{\mathcal{F}} &= \boldsymbol{\mathcal{L}}^T(\boldsymbol{\theta})\left(\boldsymbol{\mathcal{G}}\dot{\boldsymbol{\mathcal{V}}} - [\mathrm{ad}_{\boldsymbol{\mathcal{V}}}]^{\mathrm{T}}\boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{V}} + \overline{\boldsymbol{\mathcal{F}}}_{\text{tip}}\right), \\ \boldsymbol{\tau} &= \boldsymbol{\mathcal{A}}^{\mathrm{T}}\boldsymbol{\mathcal{F}} \end{aligned}$$