Ch7: Stability for Non-Autonomous Systems

Concepts of Stability

Autonomous vs. Non-Autonomous Systems

The fundamental difference between autonomous and non-autonomous systems lies in the fact that the **state trajectory of an autonomous system is independent of the initial time** t_0, while that of a non-autonomous system generally is **not**.

$$\dot{\boldsymbol{x}}(t) = \mathbf{f}(\boldsymbol{x}(t), t)$$

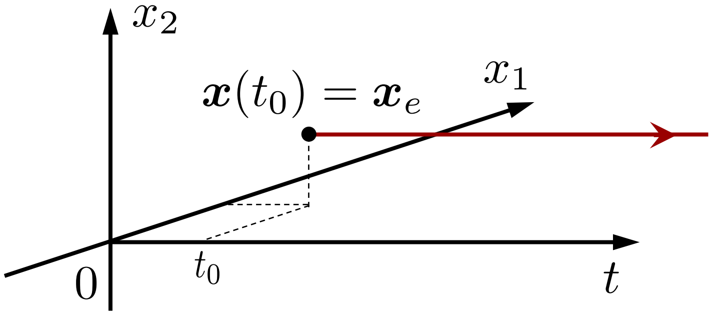
This difference requires us to **consider the initial time** t_0 **explicitly** in defining stability concepts for non-autonomous systems and makes the analysis more difficult than that of autonomous systems.

Non-autonomous systems appear in robot control when the desired task is to follow a time-varying trajectory, i.e. in **motion control**, or when there is uncertainty in the physical parameters and therefore, an **adaptive control** approach may be used.

Equilibrium Point

A state $\boldsymbol{x}_e$ is an **Equilibrium Point** (or **Equilibrium State**) if the system starts there (initial state $\boldsymbol{x}(t_0) = \boldsymbol{x}_e$) it will remain there for all future time.

$$\dot{\boldsymbol{x}} = \mathbf{f}(\boldsymbol{x}_e, t) = \mathbf{0} \qquad \forall t \geq t_0$$



For example, the system $\dot{x} = -\dfrac{a(t)x}{1 + x^2}$ has an equilibrium point at $x = 0$.

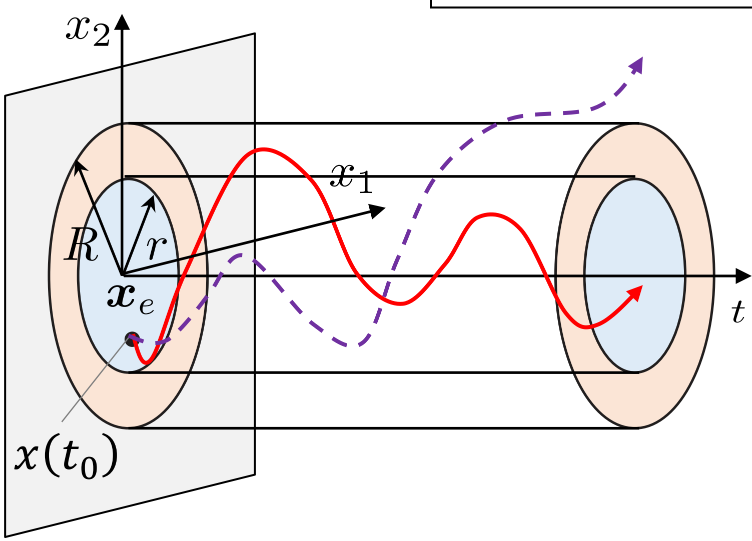
However, the system $\dot{x} = -\dfrac{a(t)x}{1 + x^2} + b(x),\ \ b(x) \neq 0$ does not have an equilibrium point.

Extensions of Stability Concepts

The concepts of stability for non-autonomous systems are quite similar to those of autonomous systems. However, the definitions include the **initial time** t_0 explicitly.

The equilibrium point $\boldsymbol{x}_e$ is said to be **Stable** at t_0 if for any $R > 0$, there exists $r = r\,(R, t_0) > 0$, such that if $\|\boldsymbol{x}\,(t_0) - \boldsymbol{x}_e\| < r$, then $\|\boldsymbol{x}\,(t) - \boldsymbol{x}_e\| < R$ for all $t \geq t_0$. Otherwise, the equilibrium point is **Unstable**.

$$\forall R > 0, \exists r > 0 : \|\boldsymbol{x}(t_0) - \boldsymbol{x}_e\| < r \Rightarrow \|\boldsymbol{x}(t) - \boldsymbol{x}_e\| < R, \ \forall t \geq t_0$$



(we can keep the state in a ball of arbitrarily small radius R by starting the state trajectory in a ball of sufficiently small radius r)

The equilibrium point $\boldsymbol{x}_e$ is said to be **Uniformly Stable**, if r can be chosen **independently** of the initial time t_0.

Extensions of Stability Concepts (cont.)

The equilibrium point $\boldsymbol{x}_e$ is said to be **Asymptotically Stable** at t_0 if (1) it is **Lyapunov Stable**, and (2) there exists $r = r(t_0) > 0$ such that if $\|\boldsymbol{x}(t_0) - \boldsymbol{x}_e\| < r$, then $\|\boldsymbol{x}(t) - \boldsymbol{x}_e\| \rightarrow 0$ as $t \rightarrow \infty$.

$$\exists r > 0 : \|\boldsymbol{x}(t_0) - \boldsymbol{x}_e\| < r \Rightarrow \|\boldsymbol{x}(t) - \boldsymbol{x}_e\| \rightarrow 0, \ \text{as } t \rightarrow \infty$$

The equilibrium point $\boldsymbol{x}_e$ is said to be **Uniformly Asymptotically Stable**, if it is **Uniformly Stable** (i.e., r can be chosen **independently** of the initial time t_0) where

$$\exists r > 0 : \|\boldsymbol{x}(t_0) - \boldsymbol{x}_e\| < r \Rightarrow \|\boldsymbol{x}(t) - \boldsymbol{x}_e\| \rightarrow 0, \ \text{as } t \rightarrow \infty$$

Example:

$$\dot{x} = -\frac{x}{(1+t)} \longrightarrow x(t) = \frac{1+t_0}{1+t} x(t_0)$$

The origin is asymptotically stable but not uniformly asymptotically stable, because a larger t_0 requires a longer time to get close to the origin.

★ Non-autonomous systems with uniform properties have some desirable ability to withstand disturbances.

Extensions of Stability Concepts (cont.)

The equilibrium point $\boldsymbol{x}_e$ is said to be **Exponentially Stable** if there exist $\alpha, \lambda, r > 0$ such that if $\|\boldsymbol{x}(t_0) - \boldsymbol{x}_e\| < r$, then $\|\boldsymbol{x}(t) - \boldsymbol{x}_e\| < \alpha \|\boldsymbol{x}(t_0) - \boldsymbol{x}_e\| e^{-\lambda(t-t_0)} \quad \forall t \geq t_0$.

$$\exists \alpha, \lambda, r > 0 : \|\boldsymbol{x}(t_0) - \boldsymbol{x}_e\| < r \Rightarrow \|\boldsymbol{x}(t) - \boldsymbol{x}_e\| \leq \alpha \|\boldsymbol{x}(t_0) - \boldsymbol{x}_e\| e^{-\lambda(t-t_0)}$$

If asymptotic (or exponential) stability holds for **any initial states** $\boldsymbol{x}(t_0) \in \mathbb{R}^n$, the equilibrium point is said to be **Globally Asymptotically (or Exponentially) Stable**.

★ It can be shown that **exponential stability** always implies **uniform asymptotic stability**.

Example: A First-Order Linear Time-varying System

Consider the first-order system $\dot{x}(t) = -a(t)x(t)$

Its solution is $x(t) = x(t_o)\mathrm{e}^{-\int_{t_0}^{t} a(r)dr}$

The system is stable if $a(t) \geq 0, \forall t \geq t_0$. It is asymptotically stable if $\int_0^\infty a(r)dr = +\infty$.

For Example:

$\dot{x} = -\frac{x}{(1+t)^2}$: The origin is stable (but not asymptotically stable), because $\int_0^\infty \frac{1}{(1+r)^2}dr = 1$.

$\dot{x} = -\frac{x}{1+t}$: The origin is asymptotically stable, because $\int_0^\infty \frac{1}{1+r}dr = +\infty$.

$\dot{x} = -tx$: The origin is exponentially stable, because $x = c_1 e^{-t^2/2}$.

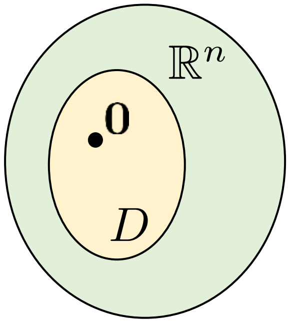
Lyapunov Analysis

Time-Varying Positive Definite Functions

A scalar, time-varying function $V(\boldsymbol{x},t)$ ($V: D \times \mathbb{R}_+ \to \mathbb{R},\ D \subset \mathbb{R}^n,\ \mathbf{0} \in D$) is said to be **Locally Positive Definite** if

1) $V(\mathbf{0},t) = 0 \qquad \forall t \geq t_0$
2) $V(\boldsymbol{x},t) \geq V_0(\boldsymbol{x}) \qquad \forall t \geq t_0\ ,\ \forall \boldsymbol{x} \in D$

where $V_0(\boldsymbol{x})$ ($V_0 : D \to \mathbb{R}$) is a **time-invariant positive definite** function.



$V(\boldsymbol{x},t)$ is said to be **Globally Positive Definite** if $D = \mathbb{R}^n$.

$\Rightarrow$ A scalar time-variant function $V(\boldsymbol{x},t)$ is positive definite if it dominates a time-invariant positive definite function.

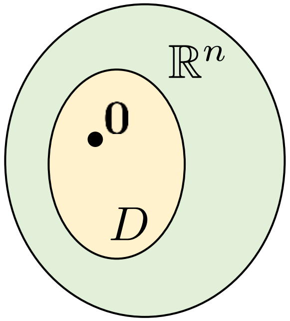
- A function $V(\boldsymbol{x},t)$ is **positive semi-definite** if $V_0(\boldsymbol{x})$ is positive semi-definite.
- A function $V(\boldsymbol{x},t)$ is **negative (semi-)definite** if $-V(\boldsymbol{x},t)$ is positive (semi-)definite.

Decrescent Function

A scalar function $V(\boldsymbol{x},t)$ ($V: D\times\mathbb{R}_+ \to \mathbb{R},\ D\subset\mathbb{R}^n,\ \mathbf{0}\in D$) is said to be **Locally Decrescent** if

1) $V(\mathbf{0},t)=0 \qquad \forall t\geq t_0$
2) $V(\boldsymbol{x},t)\leq V_1(\boldsymbol{x}) \qquad \forall t\geq t_0\ ,\ \forall \boldsymbol{x}\in D$

where $V_1(\boldsymbol{x})$ ($V_1: D\to\mathbb{R}$) is a **time-invariant positive definite** function.



$V(\boldsymbol{x},t)$ is said to be **(Globally) Decrescent** if $D=\mathbb{R}^n$.

$\Rightarrow$ A scalar time-variant function $V(\boldsymbol{x},t)$ is decrescent if it is dominated by a time-invariant positive definite function.

Example: $V(\boldsymbol{x},t)=\left(1+\sin^2 t\right)\left(x_1^2+x_2^2\right)$

$V_0(\boldsymbol{x})=x_1^2+x_2^2 \qquad V_1(\boldsymbol{x})=2\left(x_1^2+x_2^2\right)$

The function is positive definite and decrescent.

Lyapunov's Direct Method for Non-Autonomous Systems

Consider a non-autonomous system, $\dot{\boldsymbol{x}} = \mathbf{f}(\boldsymbol{x}, t)$, with an equilibrium point at origin, $\boldsymbol{x} = \mathbf{0}$. If there exists a scalar function $V(\boldsymbol{x},t)$ ($V: D \times \mathbb{R}_+ \to \mathbb{R},\ D \subset \mathbb{R}^n,\ \mathbf{0} \in D$) with continuous partial derivatives such that

1) $V(\boldsymbol{x},t)$ is **positive definite** (locally in D),
2) $\dot{V}(\boldsymbol{x},t)$ is **negative semi-definite** (locally in D),

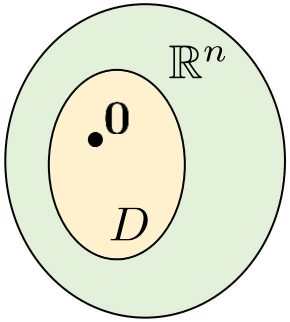
the equilibrium point $\mathbf{0}$ is **Stable** (and V is called a Lyapunov function).

3) $V(\boldsymbol{x},t)$ is **decrescent** (locally in D),

the equilibrium point $\mathbf{0}$ is **Uniformly Stable**. If $\dot{V}(\boldsymbol{x},t)$ is **negative definite** (locally in D), the equilibrium point $\mathbf{0}$ is **Uniformly Asymptotically Stable**.

4) $D = \mathbb{R}^n$,
5) $V(\boldsymbol{x},t)$ is **radially unbounded**, i.e., $V(\boldsymbol{x},t) \to \infty$ as $\|\boldsymbol{x}\| \to \infty$.

the equilibrium point $\mathbf{0}$ is **Globally Uniformly (Asymptotically) Stable**



Note:

$$\dot{V}(\boldsymbol{x},t) = \frac{dV}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial \boldsymbol{x}}\dot{\boldsymbol{x}} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial \boldsymbol{x}}\mathbf{f}(\boldsymbol{x},t)$$

Example

Example: Determine the stability of the equilibrium point at $\mathbf{0}$.

$$\dot{x}_1 = -x_1 - e^{-2t} x_2$$
$$\dot{x}_2 = x_1 - x_2$$

Let's choose this scalar function: $V(\boldsymbol{x}, t) = x_1^2 + \left(1 + e^{-2t}\right) x_2^2$

$x_1^2 + x_2^2 \le V(\boldsymbol{x}, t) \le x_1^2 + 2x_2^2$ ∴ The function is positive definite and decrescent.

$$\dot{V}(\boldsymbol{x}, t) = -2\left[x_1^2 - x_1 x_2 + x_2^2\left(1 + 2e^{-2t}\right)\right]$$

$\dot{V} \le -2\left(x_1^2 - x_1 x_2 + x_2^2\right) = -\left(x_1 - x_2\right)^2 - x_1^2 - x_2^2$ ∴ $\dot{V}$ is negative definite.

$V(\boldsymbol{x}, t)$ is radially unbounded, i.e., $V(\boldsymbol{x}, t) \to \infty$ as $\|\boldsymbol{x}\| \to \infty$.

∴ The point **0** is **globally uniformly asymptotically stable**.

Example

Consider the mass-spring-damper system

$$m\ddot{x} + c(t)\dot{x} + kx = 0$$

with time varying damping coefficient ($c(t) \geq 0$).

Physical intuition may suggest that the equilibrium point $\mathbf{0}$ is asymptotically stable as long as the damping $c(t)$ remains larger than a strictly positive constant (implying constant dissipation of energy), as is the case for autonomous nonlinear mass-spring-damper systems. **However, this is not necessarily true**.

Consider the system $\ddot{x} + (2 + e^{t})\dot{x} + x = 0$

with the initial condition $x(0) = 2$, $\dot{x}(0) = -1$, the solution is $x(t) = 1 + e^{-t}$, which tends to $x = 1$ instead! It means that the damping increases so fast that the system gets "stuck" at $x = 1$.

Stability of Linear Time-Varying Systems

Consider linear time-varying (LTV) systems of the form $\dot{\boldsymbol{x}} = \mathbf{A}(t)\boldsymbol{x}$.

LTI systems are asymptotically stable if their eigenvalues all have negative real parts. However, none of the standard approaches for analyzing LTI systems applies to LTV systems.

Example:
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & e^{2t} \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \Rightarrow \quad \lambda_{1,2} = -1, \qquad \forall t \geq 0$$

However, the system is unstable
$$x_2 = x_2(0)e^{-t}$$
$$\dot{x}_1 + x_1 = x_2(0)e^{t}$$

Stability of Linear Time-Varying Systems

The LTV system $\dot{\boldsymbol{x}} = \mathbf{A}(t)\boldsymbol{x}$ is **asymptotically stable** if the eigenvalues of the symmetric matrix $\mathbf{A}(t) + \mathbf{A}^T(t)$ (all of which are real) remain strictly in the left-half complex plane:

$$\exists\lambda > 0, \qquad \forall i, \qquad \forall t \geq 0, \qquad \lambda_i(\mathbf{A}(t) + \mathbf{A}^T(t)) \leq -\lambda$$

- Note that the result provides a **sufficient** condition for asymptotic stability.

Lyapunov-Like Analysis

Barbalat's Lemma

For autonomous systems, the invariant set theorems are powerful tools to study stability, because they allow asymptotic stability conclusions to be drawn even when $\dot{V}$ is only negative semi-definite. However, the invariant set theorems are not applicable to non-autonomous systems. Instead, Barbalat's lemma can be used for non-autonomous systems.

Barbalat's Lemma:

If the differentiable function $f(t)$ has a finite limit as $t \to \infty$, and if $\dot{f}$ is uniformly continuous, then $\dot{f}(t) \to 0$ as $t \to \infty$.

A sufficient condition for a differentiable function to be *uniformly continuous* is that its derivative be bounded.

$\Rightarrow$ If the differentiable function $f(t)$ has a finite limit as $t \to \infty$, and is such that $\ddot{f}$ exists and is bounded, then $\dot{f} \to 0$ as as $t \to \infty$.

Lyapunov-Like Stability Analysis Using Barbalat's Lemma

If a scalar function $V(\boldsymbol{x},t)$ satisfies the following conditions

- $V(\boldsymbol{x},t)$ is lower bounded,
- $\dot{V}(\boldsymbol{x},t)$ is negative semi-definite,
- $\dot{V}(\boldsymbol{x},t)$ is uniformly continuous in time (i.e., $\ddot{V}(\boldsymbol{x},t)$ is bounded),

then $\dot{V}(\boldsymbol{x},t) \to 0$ as $t \to \infty$.

Therefore, V approaches a finite limiting value V_∞, such that $V_\infty \leq V(\boldsymbol{x}(t_0),0)$.

Example

The closed-loop error dynamics of an adaptive control system for a first-order plant with one unknown parameter is

$$\dot{e} = -e + \theta w(t)$$
$$\dot{\theta} = -ew(t)$$

where e and θ are the two states of the closed-loop dynamics, representing tracking error and parameter error, and $w(t)$ is a bounded continuous function.

Consider Lyapunov function $V = e^2 + \theta^2$. The time derivative is

$$\dot{V} = 2e(-e + \theta w) + 2\theta(-ew) = -2e^2 \leq 0$$

Based on Lyapunov theory, the system is stable, and therefore, e and θ are <u>bounded</u>.

Example (cont.)

To use Barbalat's lemma, we must check the uniform continuity of $\dot{V}$.

$$\ddot{V} = -4e(-e + \theta w)$$

The derivative of $\dot{V}$ (i.e., $\ddot{V}$) is bounded, since w is bounded by hypothesis, and e and θ were shown to be bounded. Hence, $\dot{V}$ is uniformly continuous, and application of Barbalat's lemma indicates that $e \to 0$ as $t \to \infty$ ($\dot{V}(\boldsymbol{x}, t) \to 0$ as $t \to \infty$).

Note: Although e converges to zero, the system is not asymptotically stable, because θ is only guaranteed to be bounded.

Simulation with

$$w(t) = 1/(1 + t),$$
$$e(0) = \theta(0) = 0.1$$